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On supremum-norm cost games

Ana Meca, Greys Sošić *

Center for Operation Research, University Miguel Hernández, 03202 Elche, Spain
Marshall School of Business, University of Southern California, Los Angeles, CA 90089, United States

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ABSTRACT

This paper presents a new class of cooperative cost games, supremum-norm cost games, which emerges as a natural extension of k-norm games introduced by Meca and Sošić (2014) and can be seen as a generalization of the airport games introduced by Littlechild and Owen (1973). We show that it is reasonable to expect formation of the grand coalition in such setting, and describe allocations that lead to stability of the grand coalition and reduce the individual cost of each agent.

1. Introduction

Consider a setting in which a set of agents, $N = \{1, 2, \ldots, n\}$, sell an identical product sourced from the same supplier. Agents can form coalitions, $S \subset N$, and coalition members can cooperate by placing joint orders for the product. The order placed by each of the agents in the coalition will be transported to their respective warehouse, so we refer to this type of collaboration as *transportation coalitions*. We assume that there exists some a priori information about the cost of transporting an order to every agent i in coalition S, given by $c_i^S(>0)$ for $i \in S$, $S \subseteq N$. We will denote the vector of individual agents' costs in all possible subsets by $\mathbf{c}^N = (c_i^S)_{i\in S, \emptyset\neq S\subseteq N}$. Cooperation among agents is beneficial if agents' costs in larger sets do not exceed their costs in smaller ones, that is, if the agents costs in different coalitions satisfy some sort of monotonicity. In this setting, there can be several ways to measure the total cost generated by a transportation coalition $S \in N$. The simplest model assumes that the transportation cost is measured linearly, $c(S) = \sum_{i\in S} c_i^S$, and will be referred to as *a linear transportation situation*. Another intuitive model uses the Euclidean norm, $\left(\sum_{i\in S}\left(c_i^S\right)^2\right)^{1/2}$, and will be referred to as *an Euclidean transportation situation*. The third model, which can be applied when all the agents are located on the same line route, assumes that if a group of agents S places a joint order, its transportation cost is measured as the maximal distance from the provider, $c(S) = \max_{i\in S} c_i^S$. This will be referred to as *a supremum transportation situation*.

When choosing a way of measuring the cost of a coalition, the natural questions to consider are: (i) Under what conditions is it reasonable to expect that all agents will collaborate together, that is, will form the grand coalition? (ii) In the case that the grand coalition has been formed, is it always possible to find stable allocations that discourage defection of any agent? To answer the questions above, we can define a corresponding transportation game as a TU cost game (N, c), where c measures the cost generated by each of the coalitions $S \subseteq N$. Then, the above questions are equivalent to: (i) Under what conditions is the game (N, c) subadditive? (ii) If the game (N, c) is subadditive, does it have a nonempty core? [5] answer those questions for linear and Euclidean transportation situations, by means of k-norm games. In this paper, we address these questions for supremum transportation situation and find that both answers are positive.

The plan of the paper is as follows. We introduce the model in Section 2. Section 3 analyzes the core of supremum-norm cost games and describes the set of all core allocations. In Section 4 we discuss stable allocations for supremum-norm cost games. All proofs are in the technical Appendix.

2. Supremum-norm cost games

In this section, we first introduce cost-coalitional problems and some of their desirable properties, and then define supremum-norm cost games.

* Corresponding author at: Marshall School of Business, University of Southern California, Los Angeles, CA 90089, United States.
E-mail addresses: ana.meca@umh.es (A. Meca), sosic@marshall.usc.edu (G. Sošić).

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2.1. Cost-coalitional problem

Let $N = \{1, 2, \ldots, n\}$ denote the set of all agents and $S \subseteq N$ be an arbitrary set of agents in N. We assume that each member of S incurs certain positive cost, which depends on the subset S to which he belongs; we denote this cost by $c_i^S (>0)$ for $i \in S,\ S \subseteq N$. To simplify the notation, we use c_i to denote an agent's stand-alone cost, $c_i = c_i^{\{i\}}$. In addition, we denote the vector of individual agents' costs in all possible subsets by $\mathbf{c}^N = (c_i^S)_{i\in S, \emptyset\neq S\subseteq N}$, and define a *cost-coalitional problem* as a pair $(N, \mathbf{c}^N)$, where N is the set of all agents and $\mathbf{c}^N$ is the cost-coalitional vector.

Note that cooperation is beneficial only if agents' costs in larger subsets do not exceed their costs in smaller ones. This is a desirable property that we formalize below.

Property 1. *A cost-coalitional vector* $\mathbf{c}^N$ *satisfies* cost monotonicity *if each agent's cost in a given set does not exceed his cost in its subset,* $c_i^S \le c_i^T$*, for all* $i \in T,\ T \subset S$.

In the set of all agents, we want to identify a special subset. For a given cost-coalitional problem $(N, \mathbf{c}^N)$, let us denote by $\mathcal{E}(N, c)$ the set of extreme players—those with the maximum cost in the grand coalition, $\mathcal{E}(N, \mathbf{c}^N) = \{i \in N : c_i^N = \max_{j\in N} c_j^N\}$. Note that $\mathcal{E}(N, \mathbf{c}^N) \neq \emptyset$. We use e to denote cardinality of $\mathcal{E}(N, \mathbf{c}^N)$, $e = |\mathcal{E}(N, \mathbf{c}^N)|$.

2.2. Cooperative cost games

We now introduce a new class of cooperative games, supremum-norm cost games, that can be seen as a natural extension of k-norm games introduced by [5] when $k = \infty$.

First, we introduce terminology from game theory. Each set $S \subseteq N$ is referred to as a *coalition*, and $S = N$ is referred to as the *grand coalition*. Let (N, c) denote a cost game, where $c(\cdot)$ denotes the cost function defined on the set of all subsets of N.

For a given cost-coalitional problem, $(N, \mathbf{c}^N)$, we can associate a k-norm cost game. [5] define k-norm games for $k = 1, 2$, as cost games (N, c) for which $c(S)^k = \sum_{i\in S} \left(c_i^S\right)^k$. We say that a cost game (N, c) is a *supremum-norm cost game* if the cost of each coalition is obtained by taking the limit when $k \to \infty$,

$$c(S) := \lim_{k\to\infty} \left(\sum_{i\in S} \left(c_i^S\right)^k\right)^{1/k} = \max_{i\in S} c_i^S.$$

Notice that, for $1 < k < \infty$, $\max_{i\in S} c_i^S \le \left(c(S)^k\right)^{1/k} \le \sum_{i\in S} c_i^S$.

While the traditional definition of a cooperative cost game assigns a cost to each specific coalition, in our model we also know the cost of each member of a specific coalition. This additional information is useful when we study the role played by extreme players in achieving stability of the grand coalition.

Each cost-coalitional vector $\mathbf{c}^N$ leads to a unique supremum-norm cost game and vice versa. Hereafter, we focus on games whose cost-coalitional vectors satisfy monotonicity property (Property 1), and refer to them as *supremum-norm cost monotonic (SCM) games*. In the class of SCM games, we want to identify a special subclass for which $c_i^S = c_i$ for all $i \in N,\ S \subseteq N$; we will refer to the games that satisfy this property as supremum-norm fixed cost games, or *SFC games*. We can observe that the airport games, introduced in [4], are, in fact, SFC games. Another example of possible application of SCM games is cooperative purchasing. In their recent paper, [6] study cooperative purchasing situations in which each of n buyers orders inventory (an exogenous order quantity) from a common supplier, who offers a quantity discount scheme. If a subgroup of buyers is formed, the unit price paid by its members is determined by its member with the largest order quantity. While the authors of the paper focus on the savings game (the difference between the cost if a buyer acts alone or as a part of a group), we can see that the underlying cost game corresponds to a SFC game.

To complete this section, we introduce the notion of *subadditivity* and *concavity*. A game is said to be subadditive if for each $S, T \subset N$ such that $S \bigcap T = \emptyset$, it holds that $c\left(S \bigcup T\right) \le c(S) + c(T)$. Thus, as the cost of two disjoint coalitions after merger does not exceed the sum of their costs before merger, when a supremum game is subadditive, it is reasonable to expect formation of the grand coalition. Our analysis shows that this is always true for SCM games.

Proposition 1. *SCM games are subadditive.*

A game is said to be concave if for each $S, T \subset N$ it holds that $c(S \bigcup T) + c(S \bigcap T) \le c(S) + c(T)$. It can be shown that this condition is equivalent to $c\left(S \bigcup \{i\}\right) - c(S) \le c\left(T \bigcup \{i\}\right) - c(T),\ i \notin S,\ T \subset S \subseteq N$. We show that this holds for SFC games.

Proposition 2. *SFC games are concave.*

Thus, whenever the cost incurred by a player does not change in different coalitions, the resulting game is concave and has a nonempty core (see [10]). We now show that this concavity property can be extended to a larger class of games, which we call *ordered supremum-norm cost monotonic (OSCM) games*. An OSCM game is a SCM game with cost-coalitional vector $\mathbf{c}^N$ that satisfies the following properties:

- Order preservation: if $c_i^S > c_k^S$ for some $i, k \in S \subseteq N$, then $c_i^T \ge c_k^T$ for all $T \subseteq N$ such that $i, k \in T$;
- Ordered differences: if $c_i > c_k$ and $T \subset S$, then $c_k^T - c_k^S \le c_i^T - c_i^S$;
- Decreasing individual differences: for $i \in T \subset S \subseteq N$ and $V \subset N$ such that $V \cap S = \emptyset$, $c_i^T - c_i^{T\cup V} \le c_i^S - c_i^{S\cup V}$;
- Ordered decreasing differences: for $k \in T \subset S \subseteq N$, $i \in S \subseteq N$ and $V \subset N$ such that $V \cap S = \emptyset$, if $c_i > c_k$, then $c_k^T - c_k^{T\cup V} \le c_i^S - c_i^{S\cup V}$.

Order preservation implies that if one agent's cost exceeds another agent's cost in one coalition, his cost can never be lower than that of the other agent; thus, the order of agents' costs is preserved in all coalitions. Ordered differences imply that if one agent has higher cost than the other, then the cost reduction stemming from increased coalition size of the agent with higher cost cannot be smaller than the cost reduction of the agent with lower cost. Decreasing individual differences imply that if a set of agents join an existing coalition, each agent in the original coalition sees greater reduction in his cost when the original coalition is larger in size. Finally, ordered decreasing differences imply that if a set of agents joins an existing coalition, an agent with a higher cost belonging to a larger original coalition sees greater reduction in his cost than an agent with a lower cost belonging to a smaller original coalition. It is easy to verify that each SFC game is also an OSCM game. For the OSCM games, we have the following result.

Proposition 3. *OSCM games are concave.*

There exist concave SCM games that are not OSCM games; for instance, if $N = \{1, 2, 3\}$, $c_1^S = 1$ for all $S \subseteq N$, $c_2^S = 2$ for all $S \subset N$, $c_2^N = 0.5$, and $c_3^S = 3$ for all $S \subseteq N$, the resulting game is concave and does not satisfy order preservation.

Next, we study conditions for stability of the grand coalition in the sense of the core.