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Radius of robust feasibility formulas for classes of convex programs with uncertain polynomial constraints

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ABSTRACT

The radius of robust feasibility of a convex program with uncertain constraints gives a value for the maximal 'size' of an uncertainty set under which robust feasibility can be guaranteed. This paper provides an upper bound for the radius for convex programs with uncertain convex polynomial constraints and exact formulas for convex programs with SOS-convex polynomial constraints (or convex quadratic constraints) under affine data uncertainty. These exact formulas allow the radius to be computed by commonly available software.

1. Introduction

Robust convex optimization [4,3,5,11,17,16] deals with solutions of robust counterparts of uncertain convex programs where the data uncertainty is treated as deterministic, as opposed to stochastic that is used in stochastic programming. It has emerged as a powerful numerically tractable approach to treat uncertainty in convex programming. Yet, one notable limitation of its application is that the robust counterpart, where the uncertainty is enforced for every data within a specified uncertainty set, may not have a feasible solution, resulting in an infeasible robust convex program. A formula for calculating the maximal 'size' of the specified uncertainty set has long been sought so that feasibility of the robust convex program, known as robust feasibility, can be guaranteed.

In this paper, we provide such results by introducing the notion of radius of robust feasibility in robust convex optimization. It was inspired by the notion of consistency radius used in linear semi-infinite programming in order to guarantee the feasibility of the nominal problem under perturbations preserving the number of constraints [8,9,7]. This notion extends the concept of radius of robust feasibility introduced in [13] for robust linear programs.

We first derive an upper bound for the radius in the general case of convex programs with convex polynomial constraints under uncertainty both in the affine and non-affine data. We then present an exact formula for the radius of robust feasibility of a convex program with uncertain SOS-convex polynomial constraints under affine data uncertainty. In particular, we show that the radius of robust feasibility can be given in terms of the optimal value of a convex quadratic program with sum-of-squares constraints. This value can be found by solving an equivalently reformulated linear semi-definite programming (SDP in brief) problem. Thus, the radius can easily be calculated using commonly available algorithms and software. In the special case of convex programs with uncertain convex quadratic constraints under affine data uncertainty, we show that the radius of robust feasibility can be found by solving a simple explicit semi-definite linear program.

The paper is organized as follows. Section 2 provides an upper bound for the radius of robust feasibility for convex programs with uncertain convex polynomial constraints. Section 3 gives exact radius of robust feasibility formulas under affine data uncertainty for convex programs with SOS-convex polynomial constraints or convex quadratic constraints.

Notations: Before we move to the next section, we introduce some necessary notation. We denote by 0_n and $\|\cdot\|$ the vector of zeros and the Euclidean norm in $\mathbb{R}^n$, respectively. The inner product between $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$, is defined by $\langle x, y \rangle = x^T y$. The closed unit ball and the distance associated to the above norm are denoted by $\mathbb{B}_n$ and d, respectively. Given $Z \subset \mathbb{R}^n$, $\operatorname{int} Z$, $\operatorname{cl} Z$, $\operatorname{bd} Z$, and $\operatorname{conv} Z$ denote the interior, the closure, the boundary and the convex hull of

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Z, respectively, whereas cone $Z := \mathbb{R}_+ \text{conv}\, Z$ denotes the convex conical hull of $Z \cup \{0_n\}$. A symmetric $n \times n$ matrix A is said to be *positive semi-definite*, denoted by $A \succeq 0$, whenever $x^T Ax \geq 0$ for all $x \in \mathbb{R}^n$. We also use S^n_+ to denote the cone consisting of all symmetric $n \times n$ positive semi-definite matrices. The $(n \times n)$ identity matrix is denoted by I_n. Let Z be a closed and convex set in $\mathbb{R}^n$ with $0_n \in \text{int}Z$. We define a convex function $\phi_Z : \mathbb{R}^n \to \mathbb{R}_+$ by $\phi_Z(x) = \inf\{t > 0 : x \in tZ\}$. The function ϕ_Z is indeed a positive homogeneous convex function and is known as the Minkowski functional in the convex analysis literature, and is an extension of the usual norm function. In particular, if $Z = \mathbb{B}_n$, then $\phi_Z(x) = \|x\|$.

2. Uncertain convex polynomial constraints

We begin by examining a convex program with general uncertain convex polynomial constraints

$$\min_{x\in\mathbb{R}^n}\{\overline{f}(x) : \overline{g}_j(x) \leq 0,\ j \in J\}, \tag{P}$$

where $J = \{1, 2, \dots, q\}$ is a finite index set, $\overline{f} : \mathbb{R}^n \to \mathbb{R}$ is a convex function and $\overline{g}_j : \mathbb{R}^n \to \mathbb{R}$ is a convex polynomial. The robust counterpart of the uncertain convex program (P) is given by

$$\min_{x\in\mathbb{R}^n}\left\{\overline{f}(x) : \overline{g}_j(x) + \sum_{l=1}^{p} v_j^l \overline{g}_j^l(x) + a_j^T x + b_j \leq 0,\ \forall\big(v_j, (a_j, b_j)\big) \in \bar{\alpha}_j\big(M \times \mathbb{B}_{n+1}\big),\ j \in J\right\}, \tag{RP$_{\bar{\alpha}}$}$$

where $\bar{g}_j^l$ are convex polynomials on $\mathbb{R}^n$, $l = 1, \dots, p$, $v_j = \big(v_j^1, \dots, v_j^p\big) \in \mathbb{R}^p$, $M \subset \mathbb{R}^p_+$ is a convex compact set with $0_p \in M$ and $\bar{\alpha}_j \geq 0$, $j \in J$. We assume throughout this section that

$$\left\{x \in \mathbb{R}^n : \overline{g}_j(x) + \sum_{l=1}^{p} v_j^l \overline{g}_j^l(x) + a_j^T x + b_j \leq 0,\ \forall\big(v_j, (a_j, b_j)\big) \in \bar{\alpha}_j\big(M \times \mathbb{B}_{n+1}\big),\ j \in J\right\} \neq \emptyset.$$

As convexity is preserved for only nonnegative perturbations of nonlinear convex polynomials, we require that $v_j \in \mathbb{R}^p_+$, $j \in J$. Moreover, it has also been noted in [14] that if this nonnegative restriction of M is dropped, the corresponding robust optimization problem is in general NP-hard, even when $\overline{f}$ and $\overline{g}_j$ are all convex quadratic functions.

On the other hand, it is known that in the case where $\overline{f}$ and $\overline{g}_j$ are convex quadratic functions, and $M := \mathbb{B}_p \cap \mathbb{R}^p_+$ (the so-called restricted ellipsoidal uncertainty set), the optimal value of (RP$_{\bar{\alpha}}$) can be found by solving a semi-definite programming problem (see [14] and for an extension to a class of convex polynomial programs, see [20]). Various computationally tractable classes of robust counterparts of the form (RP$_{\bar{\alpha}}$) in the more general case, where $\overline{f}$ and $\overline{g}_j$ are convex polynomials, are also given in [20].

Now, consider the family of robust counterparts of the original problem (P):

$$\min_{x\in\mathbb{R}^n}\left\{\overline{f}(x) : \overline{g}_j(x) + \sum_{l=1}^{p} v_j^l \overline{g}_j^l(x) + a_j^T x + b_j \leq 0,\ \forall\big(v_j, (a_j, b_j)\big) \in (\overline{\alpha}_j + \alpha_j)\big(M \times \mathbb{B}_{n+1}\big),\ j \in J\right\}, \tag{RP$_{\bar{\alpha},\alpha}$}$$

where $\alpha_j \geq 0, j \in J = \{1, 2, \dots, q\}$ and $\overline{\alpha} = (\overline{\alpha}_1, \dots, \overline{\alpha}_q)$.

Definition 2.1 (*Radius of Robust Feasibility*)**.** The *radius of robust feasibility*, $\rho(\overline{g}, \overline{\alpha})$, associated to $\overline{g} = (\overline{g}_1, \dots, \overline{g}_q)$ and $\overline{\alpha} \geq 0$, is defined to be

$$\rho(\overline{g}, \overline{\alpha}) := \sup\left\{\min_{j\in J} \alpha_j : (\text{RP}_{\bar{\alpha},\alpha}) \text{ is feasible}\right\}. \tag{1}$$

It is interesting to note that in the case where $\bar{\alpha} = 0_q$ the radius $\rho(\overline{g}, \overline{\alpha})$ provides the maximal size of the ball uncertainty set under which robust feasibility of (P) is guaranteed.

Recall that an extended real-valued function h on $\mathbb{R}^n$ is called proper if $h(x) > -\infty$ for all $x \in \mathbb{R}^n$ and there exists $x_0 \in \mathbb{R}^n$ such that $h(x_0) < +\infty$. Denote by $\Gamma(\mathbb{R}^n)$ the class of proper convex lower semicontinuous (lsc) extended real-valued functions. Now let $h \in \Gamma(\mathbb{R}^n)$. The effective domain and the epigraph of h are defined respectively as follows:

$$\text{dom}\, h = \{x \in \mathbb{R}^n : h(x) < +\infty\} \quad \text{and}$$
$$\text{epi} h = \{(x, \gamma) \in \mathbb{R}^n \times \mathbb{R} : x \in \text{dom}\, h,\ h(x) \leq \gamma\}.$$

The conjugate function of h, $h^* : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, is defined by $h^*(v) = \sup\{v^T x - h(x) : x \in \text{dom}\, h\}$. Note that, for $h_1, h_2 \in \Gamma(\mathbb{R}^n)$,

$$\text{epi}(h_1 + h_2)^* = \text{epi} h_1^* + \text{epi} h_2^*, \tag{2}$$

provided either h_1 or h_2 is a real-valued convex function. Moreover, for a proper lsc convex function h, we have

$$\text{epi}(\alpha h)^* = \alpha\, \text{epi} h^* + \big(\{0_n\} \times \mathbb{R}_+\big) \quad \text{for } \alpha \geq 0. \tag{3}$$

The following result plays a key role in the next section in developing upper bounds for the radius of robust feasibility.

Lemma 2.1 (*[10]*)**.** *Let $h_t \in \Gamma(\mathbb{R}^n)$ for all $t \in T$ (an arbitrary index set). Then, $\{x \in \mathbb{R}^n : h_t(x) \leq 0, t \in T\} \neq \emptyset$ if and only if $(0, -1) \notin \text{cl cone}\big(\bigcup_{t\in T} \text{epi} h_t^*\big)$.*

In order to establish bounds for the radius of robust feasibility, we need the following lemma.

Lemma 2.2. *Let $h_j : \mathbb{R}^n \to \mathbb{R}, j \in J$, be convex functions. Let $\beta \geq 0$, and let $Z_0 \subset \mathbb{R}^{n+1}$ be a compact and convex set with $0_{n+1} \in \text{int}Z_0$. Suppose that $(0_n, -1) \in \text{cl cone}\big(\bigcup_{j\in J} \text{epi} h_j^* + \beta Z_0\big)$. Then, for all $\delta > 0$, we have $(0_n, -1) \in \text{cone}\big(\bigcup_{j\in J} \text{epi} h_j^* + (\beta + \delta)Z_0\big)$.*

Proof. We proceed by the method of contradiction and assume that there exists $\delta > 0$ such that $(0_n, -1) \notin \text{cone}\big(\bigcup_{j\in J} \text{epi} h_j^* + (\beta + \delta)Z_0\big)$. Then, the separation theorem implies that there exists $(\xi, r) \in \mathbb{R}^{n+1} \setminus \{0_{n+1}\}$ such that for all $(y, s) \in \text{cone}\big(\bigcup_{j\in J} \text{epi} h_j^* + (\beta + \delta)Z_0\big)$,

$$-r = \langle(\xi, r), (0_n, -1)\rangle \leq 0 \leq \langle(\xi, r), (y, s)\rangle. \tag{4}$$

Note that $(0_n, -1) \in \text{cl cone}\big(\bigcup_{j\in J} \text{epi} h_j^* + \beta Z_0\big)$. As $|J| < +\infty$, by passing to subsequence if necessary, the Carathédory's Theorem implies that there exist $j_l \in J$, $l = 1, \dots, n+2$, $\{\mu_k^{j_l}\} \subset \mathbb{R}_+$, $\{u_k^{j_l}\} \subset \text{dom} h_{j_l}^*$, $\{\epsilon_k^{j_l}\} \subset \mathbb{R}_+$ and $(z_k^{j_l}, t_k^{j_l}) \subset Z_0$, such that

$$(y_k, s_k) := \sum_{l=1}^{n+2} \mu_k^{j_l}\Big(\big(u_k^{j_l}, h_{j_l}^*(u_k^{j_l}) + \epsilon_k^{j_l}\big) + \beta(z_k^{j_l}, t_k^{j_l})\Big) \to (0_n, -1) \quad \text{as } k \to \infty.$$