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Analytical representations for the basic affine jump diffusion

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ABSTRACT

The Basic Affine Jump Diffusion (BAJD) process is widely used in financial modeling. In this paper, we develop an exact analytical representation for its transition density in terms of a series expansion that is uniformly-absolutely convergent on compacts. Computationally, our formula can be evaluated to high level of accuracy by easily adding new terms which are given explicitly. Furthermore, it can be easily generalized to give an analytical expression for the transition density of the subordinate BAJD process which is more realistic than the BAJD process, while existing approaches cannot.

1. Introduction

We consider the *Basic Affine Jump Diffusion (BAJD)* process introduced in Duffie and Gârleanu [13], which is the unique strong solution to the following stochastic differential equation:

$$dX_t = \kappa(\theta - X_t)dt + \sigma\sqrt{X_t}dB_t + dJ_t, \quad X_0 = x \geq 0.$$

Here $\kappa, \theta, \sigma > 0$ are the rate of mean reversion, the long-run mean, and the volatility coefficient, respectively. $J := (J_t)_{t\geq 0}$ is a compound Poisson process with arrival rate $\varpi \geq 0$, and its jumps are exponentially distributed with mean $\mu > 0$. When the Feller condition $2\kappa\theta \geq \sigma^2$ is satisfied, zero is an unattainable boundary and the state space of this process, denoted by E, is given by $E = (0, \infty)$ (Cheridito et al. [8]). If $0 < 2\kappa\theta < \sigma^2$, the process is instantaneously reflected at zero and $E = [0, \infty)$. When $\varpi \equiv 0$ (i.e., $J \equiv 0$) the BAJD process reduces to the Cox, Ingersoll, and Ross [9] (CIR) process.

The BAJD process has found many applications in finance. For instance, it is used to model the default intensity in credit risk applications (see, e.g., Duffie and Gârleanu [13], Mortensen [24], Brigo and El-Bachir [5,6], and Eckner [15]), the short-rate process in interest rate markets (see, e.g., Brigo and Mercurio [7]) and the volatility of an asset (see, e.g., Duffie et al. [14], Eraker et al. [17], and Eraker [16]). In energy markets, the BAJD process is used as the background process for modeling the spot price of electricity (Li et al. [21]).

In these applications, one is interested in computing expectations of the form

$$\mathcal{P}_t^\alpha f(x) := \mathbb{E}_x\left[e^{-\alpha\int_0^t X_u du} f(X_t)\right] \quad (\alpha \geq 0).$$

In financial terms, f is the payoff function and X is the default intensity factor in credit risk applications, or the short rate in interest rate models, or the spot price (in such case we are concerned with $\alpha = 0$). The collection of the operators $(\mathcal{P}_t^\alpha)_{t\geq 0}$ forms a Feynman–Kac (FK) semigroup of contractions on $\mathcal{B}_b(E)$, the space of Borel-measurable and bounded functions on E. The kernel of the BAJD semigroup is Sub-Markovian as $\mathcal{P}_t^\alpha 1 \leq 1$. It is absolutely continuous w.r.t. the Lebesgue measure and we denote its density by $p^\alpha(t, x, y)$, i.e.,

$$\mathcal{P}_t^\alpha f(x) = \int_E f(y)p^\alpha(t, x, y)dy. \tag{1.1}$$

Once $p^\alpha(t, x, y)$ is known, the integral in (1.1) can be obtained either analytically or numerically.

In spite of the extensive use of the BAJD process in applications, to our best knowledge, $p^\alpha(t, x, y)$ is unknown in any analytical form. In the literature, there exist two approaches for computing $p^\alpha(t, x, y)$. It is well known that the Laplace transform is given by (cf. Duffie and Gârleanu [13])

$$\mathcal{P}_t^\alpha e^{-zx} = \mathbb{E}_x\left[e^{-\alpha\int_0^t X_u du} e^{-zX_t}\right] = C(\varpi, \alpha, z; t)D(\varpi, \alpha; t) \times A(\alpha, z; t)\exp\{-B(\alpha, z; t)x\}, \quad z, \alpha \geq 0 \tag{1.2}$$

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where

$$A(\alpha,z;t) := \left(\frac{2\varepsilon e^{(\kappa+\varepsilon)t/2}}{2\varepsilon+(\varepsilon+\kappa+z\sigma^2)(e^{\varepsilon t}-1)}\right)^{\mathfrak{b}},$$

$$B(\alpha,z;t) := \frac{2\alpha(e^{\varepsilon t}-1)+z(\varepsilon-\kappa)e^{\varepsilon t}+z(\varepsilon+\kappa)}{2\varepsilon+(\varepsilon+\kappa+z\sigma^2)(e^{\varepsilon t}-1)},$$

$$C(\varpi,\alpha,z;t) := \left(1+\frac{(e^{\varepsilon t}-1)\big(\varepsilon+\kappa+z\sigma^2+\mu(2\alpha+z(\varepsilon-\kappa))\big)}{2\varepsilon(1+z\mu)}\right)^{-\varpi\mathfrak{a}}, \tag{1.3}$$

$$D(\varpi,\alpha;t) := \exp\left\{-\varpi\left(\frac{\kappa+\varepsilon}{2\varepsilon}\right)\left(\frac{\mathfrak{b}}{\mathfrak{b}-1}\right)t\right\},$$

with

$$\mathfrak{a} := \frac{2\mu}{\sigma^2-2\mu\kappa-2\alpha\mu^2}, \qquad \mathfrak{b} := \frac{2\mu\varepsilon}{\sigma^2+\mu(\varepsilon-\kappa)}, \tag{1.4}$$

$$\varepsilon := \sqrt{\kappa^2+2\alpha\sigma^2}, \quad \text{and} \quad b := \frac{2\kappa\theta}{\sigma^2}.$$

The formula can be obtained following the theory of affine processes (Duffie et al. [12]) to solve the corresponding generalized Riccati equation. Thus one approach to obtain $p^\alpha(t,x,y)$ is to invert the Laplace transform numerically. The other approach approximates the transition density either by polynomial approximations (Filipovic et al. [18]) or by approximations of the Kolmogorov forward/backward PIDE (Yu [31]).

In this paper, we derive an exact analytical expression for $p^\alpha(t,x,y)$ in terms of multiple infinite series which are uniformly-absolutely convergent on compacts. A series $\sum_{n=0}^{\infty} f_n(x)$ is said to converge *uniformly-absolutely convergent* if $\sum_{n=0}^{\infty}|f_n(x)|$ converges uniformly (a series of functions satisfying the Weierstrass's criterion for uniform convergence is uniformly-absolutely convergent, see, e.g., Itô [19], Definition 435.A, p.1647). As a by-product of our result for $p^\alpha(t,x,y)$, we also obtain the stationary density of the BAJD process.

In general, when the Laplace transform of a function is known, the function can be recovered from Laplace inversion via the Bromwich integral. In our case, we first derive an alternative representation for the Laplace transform $\mathcal{P}_t^\alpha e^{-zx}$ based on the spectral representation of the FK semigroup of the CIR process (Cox et al. [9]) and the binomial expansion. This representation allows us to calculate the Laplace inversion analytically.

To implement the existing closed-form approximations, one typically first fixes the number of terms to be used and then uses symbolic computational software to obtain the formula for these terms. Once the formula is obtained and stored, the subsequent evaluation at given parameter values can be done instantaneously. However, a potential drawback is that, one usually does not know a priori how many terms need to be used to achieve a certain level of accuracy, and adding a new term that has not been pre-computed can be costly. In contrast, in our expansion, every term is given explicitly and one can easily add a new term if it is needed to improve accuracy. In Section 4, we compare the approximation developed in Filipovic et al. [18] with our method, and it will be shown that the approximation formula which uses the first two to four terms can have quite significant error. Another nice feature of our method is that it can be easily generalized after subordination while the existing approaches cannot. The BAJD process is quite unrealistic in that it can only jump upward. Applying subordination to it allows us to develop more realistic models with two-sided jumps that are mean-reverting (see e.g., Boyarchenko and Levendorskii [4], Lim et al. [22], Mendoza-Arriaga and Linetsky [23] for applications of subordination to other processes in finance). Fig. 1 illustrates typical sample paths for (*a*) the CIR process $\widetilde{X}$, (*b*) the BAJD process X, and (*c*) the Subordinate BAJD (SubBAJD) process Y. All three processes are mean reverting, the BAJD process exhibits only positive jumps, while the SubBAJD process exhibits mean reverting (positive and negative) jumps without leaving the state space E.

The rest of the paper is organized as follows. In Section 2, we obtain analytical representations for $p^\alpha(t,x,y)$ and $\mathcal{P}_t^\alpha f(x)$. In Section 3, we extend these results to the case with subordination. Section 4 presents numerical examples. All proofs are collected in the Appendix.

2. Analytical formula for $p^\alpha(t,x,y)$

We make the following important observation: when $\varpi = 0$, since the BAJD process becomes the CIR process and $C(0,\alpha,z;t) = 1$, $D(0,\alpha;t) = 1$, the term $A(\alpha,z;t)\exp\{-B(\alpha,z;t)x\}$ is the Laplace transform of the CIR process. Hence we can rewrite Eq. (1.2) as

$$\mathcal{P}_t^\alpha e^{-zx} = C(\varpi,\alpha,z;t)D(\varpi,\alpha;t)\,\widetilde{\mathcal{P}}_t^\alpha e^{-zx},$$
$$x\in E,\ z,\alpha,t\geq 0$$

where $\widetilde{\mathcal{P}}^\alpha = (\widetilde{\mathcal{P}}_t^\alpha)_{t\geq 0}$ is the FK semigroup of the CIR process with killing rate αx. The FK semigroup of the CIR process can be represented by an eigenfunction expansion for functions that belong to $L^2(E,\mathfrak{m})$ where $\mathfrak{m}(dx) = \mathfrak{m}(x)dx$ is the CIR's speed measure with its density given by $\mathfrak{m}(x) = \frac{2x^{b-1}}{\sigma^2}e^{-2\kappa x/\sigma^2}$. Hence, from Proposition 9 in Davydov and Linetsky [10], for all $f \in L^2(E,\mathfrak{m})$ we have

$$\tilde{\mathcal{P}}_t^\alpha f(x) = \mathbb{E}_x[e^{-\alpha\int_0^t X_u du}f(X_t)] = \sum_{n=0}^{\infty} c_n e^{-\lambda_n t}\varphi_n(x),$$

$$c_n = \int_E f(x)\varphi_n(x)\mathfrak{m}(x)dx, \tag{2.5}$$

where for $n = 0, 1, \ldots,$

$$\lambda_n = n\varepsilon + \frac{b}{2}(\varepsilon-\kappa),$$

$$\varphi_n(x) = N_n^\alpha e^{((\kappa-\varepsilon)x)/\sigma^2}L_n^{b-1}\left(\frac{2x\varepsilon}{\sigma^2}\right), \tag{2.6}$$

$$N_n^\alpha = \sqrt{\frac{\sigma^2 n!}{2\Gamma(b+n)}}\left(\frac{2\varepsilon}{\sigma^2}\right)^{b/2},$$

with the variables b and ε defined in (1.4), and where $L_n^\nu(x)$ are the generalized Laguerre polynomials. It is straightforward to verify that for all $z \geq 0$, $e^{-zx} \in L^2(E,\mathfrak{m})$, hence we can calculate the Laplace transform of the CIR process using eigenfunction expansions. In particular, the expansion coefficients c_n entering into the expansion (2.5) for the function $f(x) = e^{-zx}$, $z \geq 0$, are available in close form (the calculation details are omitted), and they are given by

$$c_n(z) = \frac{1}{N_n^\alpha}\left(\frac{\kappa-\varepsilon+\sigma^2 z}{\kappa+\varepsilon+\sigma^2 z}\right)^n\left(\frac{2\varepsilon}{\kappa+\varepsilon+\sigma^2 z}\right)^b. \tag{2.7}$$

Lemma 1. *The spectral expansion (2.5) for the function $f(x) = e^{-zx}$, is uniformly-absolutely convergent on compacts for x, z and t.*

The function $C(\varpi,\alpha,z;t)$ can also be expanded in series such that time t enters the expression in an exponential form.

Lemma 2. *Define $Q(z) := \left(\frac{1}{\mathfrak{b}} - \frac{1}{\mathfrak{a}\varepsilon}\left(\frac{1}{\mu z+1}\right)\right)$ with $\mathfrak{a}$, $\mathfrak{b}$ and ε defined as in (1.4). Then, the function $C(\varpi,\alpha,z;t)$ of Eq. (1.3) accepts the*