Operations Research Letters 44 (2016) 129–135

Contents lists available at ScienceDirect

Operations Research Letters

journal homepage: www.elsevier.com/locate/orl

A note on the never-early-exercise region of American power exchange options

Daniel Wei-Chung Miao [a,b,*], Xenos Chang-Shuo Lin [c], Steve Hsin-Ting Yu [d]

[a] National Taiwan University of Science and Technology, Taipei 106, Taiwan
[b] Imperial College Business School, London SW7 2AZ, UK
[c] Aletheia University, No.32, Zhenli St., Danshui Dist., New Taipei City, 25103, Taiwan
[d] Taiwan Futures Exchange, 14th Fl., 100 Roosevelt Road, Sec. 2, Taipei, 100, Taiwan

ARTICLE INFO

Article history:
Received 21 May 2014
Received in revised form
13 December 2015
Accepted 13 December 2015
Available online 19 December 2015

Keywords:
American option
Power exchange option
Early exercise
Never-early-exercise region
Analytical upper bound
Geske–Johnson method

ABSTRACT

This note discusses how the never-early-exercise region of American power exchange options is influenced by the nonlinearity from its power coefficients. We consider a class of models which satisfy the *power invariant property* and show that early exercise depends crucially on the quantities termed *effective dividend yields*. Our mathematical analysis extends an existing model-free result and indicates how early exercise should depend on parameters. A numerical analysis is conducted to complement the analytical results and provide further observations.

1. Introduction

An exchange option gives its holder the right to exchange one asset for another. Earlier studies can be dated back to Fisher [3] and Margrabe [9] who derived the closed-form pricing formula for a European exchange option under the classical Black–Scholes model. Lindset [7] extended the pricing formula to Merton's jump–diffusion model [10] and proposed to use the Geske–Johnson method [4] to price its American version. One extension from the plain-vanilla exchange option is the power exchange option where nonlinear dependence is introduced by its power coefficients. As seen in Johnson and Tian [6] and Blenman and Clark [1], power exchange options provide more flexibility in the design of indexed executive stock options. Under the Black–Scholes model, the price of its European version was derived in closed form in [1]. By using the martingale property of the underlying stock prices, [1] also gave a sufficient condition under which its American version should never be exercised early.

The merit of this sufficient condition for the never-early-exercise (NEE) property is that it is model-free. However, the fact that it applies to a wide range of models also makes it a conservative condition. In fact, there are plenty of option parameters which actually lead to NEE but cannot be identified by this condition. In this note we consider a specific class of stock price models and show that the sufficient condition for NEE can be considerably weakened such that much more option parameters can be identified as never-early-exercise. In the model class of interest, we assume the *power invariant property* holds, meaning that the powered process of stock price remains in the same family as the original stock price process. This property enables us to introduce the *power martingale condition* and define the *effective dividend yields* which play important roles in the analysis of early exercise. A number of popular stock price models belong to this model class, including the Black–Scholes models, jump–diffusion models [10], and variance gamma models [8]. In fact, it contains all the exponential Levy models. A commonly used model not satisfying the power invariant property is the Heston stochastic volatility model [5].

When the sufficient condition for NEE is met, the American power exchange option price must be equal to the price of its European version. Contrarily, if the condition is not satisfied, early exercise may be possible and it is of interest to discuss the value contributed by early exercise. Taking the perspective of effective dividend yields, the pricing problem can be reduced to its plain-vanilla version except that the dividend yields are no longer

* Corresponding author at: National Taiwan University of Science and Technology, Taipei 106, Taiwan.
E-mail address: miao@mail.ntust.edu.tw (D.W.-C. Miao).

http://dx.doi.org/10.1016/j.orl.2015.12.011
0167-6377/

nonnegative. This gives rise to some new results which cannot be seen in the plain-vanilla setting. We derive upper bounds on the American power exchange option price and show that they take different forms according to the signs of effective dividend yields. These results enable us to see in what way the real-valued effective dividend yields affect the value and likelihood of early exercise. Moreover, they provide some computational implications for the pricing of American power exchange options. When an option's parameters are in the NEE region (condition for NEE is met), it can be valued by standard European option pricing methods. When the parameters are outside the NEE region, the upper bounds (which may be evaluated in the same way as European options) provide an indication of whether it is worth incurring the computational cost of accurately pricing the American version.

To investigate the contribution of early exercise in a more accurate way, we apply the Geske–Johnson method [4,7] to conduct a numerical analysis which provides further observations. We find that the numerical NEE region generally covers an even wider range of option parameters than the theoretical condition would suggest. Through our numerical examples, we provide in-depth discussions on how early exercise is influenced by the two power coefficients.

2. Mathematical analysis

Consider the power exchange option which gives the payoff $(S_{1t}^{n_1} - S_{2t}^{n_2})^+$ where S_{1t}, S_{2t} are stock price processes and the power coefficients n_1, n_2 are positive real numbers. The vanilla exchange option corresponds to the special case $n_1 = n_2 = 1$. Denote the option maturity time as T and let $r, q_1, q_2 \geq 0$ respectively stand for the nonnegative interest rate and the dividend yields of the two stock price processes. For the market to be free of arbitrage, the stock prices must satisfy the martingale condition

$$\mathsf{E}_t[S_{iT}] = S_{it} e^{(r-q_i)(T-t)}, \quad i = 1, 2, \tag{1}$$

for $t \leq T$ under the risk-neutral measure. It simply means that $e^{-(r-q_i)t} S_{it}$ is a martingale and this should hold for any stock price model.

2.1. The never-early-exercise conditions

Based on the fundamental relation (1), [1] gave a sufficient condition (Theorem 4, p.104) for the American power exchange option to be never-early-exercise. The result is rephrased in our notation as below.

Theorem 2.1. *Consider the American power exchange option with power coefficients n_1, n_2. If $n_1 \geq 1$ (such that x^{n_1} is a convex function of x), $0 < n_2 \leq 1$ (such that x^{n_2} is a concave function of x), and $n_1(r - q_1) \geq r$, $n_2(r - q_2) \leq r$, then early exercise is not optimal, and its value is the same as that of the European version.*

The main idea behind this theorem is that a convex function of a martingale (say $X_t = e^{-(r-q_i)t} S_{it}$) is a submartingale ($\mathsf{E}_t[X_T^{n_1}] \geq X_t^{n_1}$) while a concave function of a martingale is a supermartingale ($\mathsf{E}_t[X_T^{n_2}] \leq X_t^{n_2}$). As no specific assumption is made on the stock price model (except that r, q_1, q_2 are constant), this sufficient condition for NEE is model-free. We intend to show that for specific models, this condition can be considerably weakened such that more option parameters may lead to NEE. Before we proceed, it is worth noting that the volatilities (in a wide sense, and may include contributions from the diffusion and jump parts) of both stock price processes are absent in Theorem 2.1. This is natural since the condition is model-free whereas volatilities are model specific parameters. As will be seen later, they become present in our weakened condition for a certain class of models.

To proceed, let us consider a class of stock price processes which satisfy the following *power invariant property*.

Definition 2.2. The stock price processes $S_{it}, i = 1, 2$ are said to be power invariant if for a pair of positive $n_i, i = 1, 2$, the powered processes $S_{it}^{n_i}, i = 1, 2$ remain in the same family as the original processes $S_{it}, i = 1, 2$.

Because (1) must hold for the stock prices S_{it} (under the risk-neutral measure), one natural consequence of the above definition is that the following *power martingale condition* must hold for their powered processes

$$\mathsf{E}_t[S_{iT}^{n_i}] = S_{it}^{n_i} e^{(r-Q_i)(T-t)}, \quad i = 1, 2, \tag{2}$$

where $Q_i \in \mathbb{R}$ is called the *effective dividend yield* of $S_{it}^{n_i}$. Unlike dividend yield q_i which is nonnegative, here Q_i is an artificial quantity and can take real values. This condition was first introduced in [11] for the discussion of single asset power options. As it turns out, this condition is essential in the analysis of power exchange options.

The explicit formulas of effective dividend yields under these specific models can be derived without difficulty. Take the bivariate Black–Scholes model for example:

$$dS_{it} = (r - q_i)S_{it}dt + \sigma_i S_{it} dW_{it}, \quad i = 1, 2, \tag{3}$$

where $W_{it}, i = 1, 2$ are two standard Brownian motions with correlation ρ(i.e. $dW_{1t}dW_{2t} = \rho dt$). Using the result $\mathsf{E}[S_{iT}^{n_i}] = S_{it}^{n_i} e^{[n_i(r-q_i)+\frac{n_i(n_i-1)}{2}\sigma_i^2](T-t)}$ and matching it to (2), one obtains

$$Q_i = (1 - n_i)r + n_i q_i - \frac{n_i(n_i - 1)}{2}\sigma_i^2. \tag{4}$$

From (4), we see the role of volatility σ_i in Q_i. Specifically, under proper conditions (e.g. $n_i > 1$, r large and q_i small), a greater σ_i may lead to quite negative value of Q_i. Similar observation can be made under some other models (with Q_i formulas given in [11]) in that more volatile processes tend to make Q_i more negative (when $n_i > 1$).

Based on the notion of effective dividend yields, our weakened version of the NEE condition for this class of models is presented as follows.

Theorem 2.3. *Consider an American power exchange option with power coefficients n_1 and n_2. Suppose that the stock price processes S_{1t} and S_{2t} are power invariant with the effective dividend yields Q_1 and Q_2 well defined by (2) (i.e. $Q_i = r + \frac{1}{T-t}\ln(S_{it}^{n_i}/\mathsf{E}[S_{iT}^{n_i}])$, $i = 1, 2$, which do not depend on S_{it} or $T - t$). If $Q_1 \leq 0$ and $Q_2 \geq 0$, then early exercise is never optimal, and its value is the same as that of its European version.*

Proof. We prove the claim by showing that the price of European power exchange (EPE) option is always higher than the exercise value of its American counterpart, regardless of how high S_{1t} is or how low S_{2t} is. This is seen from

$$\begin{aligned}
\text{EPE price} &= e^{-r(T-t)}\mathsf{E}_t[(S_{1T}^{n_1} - S_{2T}^{n_2})^+] \\
&\geq e^{-r(T-t)}(\mathsf{E}_t[S_{1T}^{n_1}] - \mathsf{E}_t[S_{2T}^{n_2}])^+ \quad (\because \text{Jensen's inequality}) \\
&= e^{-r(T-t)}(S_{1t}^{n_1}e^{(r-Q_1)(T-t)} - S_{2t}^{n_2}e^{(r-Q_2)(T-t)})^+ \\
&= (S_{1t}^{n_1}e^{-Q_1(T-t)} - S_{2t}^{n_2}e^{-Q_2(T-t)})^+ \geq (S_{1t}^{n_1} - S_{2t}^{n_2})^+,
\end{aligned}$$

where the conditions $Q_1 \leq 0$ and $Q_2 \geq 0$ are used to ensure the last inequality holds. □

To show how the NEE condition is weakened, we take the Black–Scholes model for example. Let q_i^* stand for the solution of $Q_i = 0$ with Q_i defined in (4), i.e.

$$q_i^* = \frac{n_i - 1}{n_i}r + \frac{n_i - 1}{2}\sigma_i^2, \quad i = 1, 2. \tag{5}$$

Table 1 provides a comparison between the sufficient conditions in Theorems 2.1 and 2.3 which are equivalently expressed in terms