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Some specially structured assemble-to-order systems

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ABSTRACT

Assemble-to-order systems are important in practice but challenging computationally. This paper combines some notions from combinatorial optimization, namely polymatroids and discrete convexity, to ease the computational burden significantly, for certain specially structured models. We point out that polymatroids have a concrete, intuitive interpretation in this context.

1. Introduction

This paper considers a one-period *assemble-to-order* (or *ATO*) *system* with stochastic demand. There are several *components* and several *products*. Demands occur only for products, while the system keeps inventory only of components. To make a unit of each product requires a certain amount of each component. The time to assemble a product from components is negligible. A product is assembled only in response to demand. The components must be acquired, however, before demand is realized. The overall problem can be viewed as a stochastic program with recourse. See Song and Zipkin [31] for a review of the research literature and applications up to the early 2000's. We mention several more recent works below.

Once demand is realized, the available components must be allocated among the products. This problem can be represented as an integer linear program (ILP). For certain product-component structures, under a certain condition on the component inventories, the feasible set has a special form called a *polymatroid*. One implication of this form is that the ILP is easy to solve; its solution can be written almost in closed form.

We prove that, assuming only that the acquisition cost is increasing, the component inventories *should* satisfy a related but weaker condition. (More precisely, this is a necessary condition for optimality in the larger problem of component acquisition.) In certain special but important cases, this weaker condition is in fact equivalent to the polymatroid condition. In any case, we assume the polymatroid condition from then on.

Next, we show that the expected cost of the ILP has a property related to $L^{\natural}$*-convexity*. This notion, developed by Murota [22,23] and others, is part of a broader topic called discrete convexity. It has proven quite useful in understanding and solving other operational problems. (The last section below mentions some of these.) Now, the cost here is not quite $\mathrm{L}^{\natural}$-convex; it has a property we call *cover-*$L^{\natural}$*-convexity*. We suggest that the methods developed in the discrete-convexity literature may be adapted to this case. (Caveat: We have not worked out the details.) If so, provided the acquisition cost also is well behaved, the overall problem should be fairly tractable.

Preliminary numerical results indicate that the polymatroid condition in fact often holds, even when it is not guaranteed. Otherwise, when it is imposed as a constraint, the cost penalty is fairly small.

The polymatroid condition may seem abstract. We point out, however, that it has a rather appealing, concrete meaning in the context of ATO systems.

Section 2 reviews relevant material on polymatroids and discrete convexity, derives a few small but useful results about polymatroids, and introduces the notion of cover-$\mathrm{L}^{\natural}$-convexity. Section 3 presents the formulation of the ATO model. Section 4 contains the main results concerning the relations of the ATO model with polymatroids and cover-$\mathrm{L}^{\natural}$-convexity. Section 5 reports the numerical results. Section 6 extends the model and results to two multi-period systems, one with backorders and one with lost sales (but zero leadtimes). Finally, Section 7 points out some directions for further study, while suggesting some intuitive interpretations.

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2. Preliminaries

2.1. Polymatroids

2.1.1. Basics

The real numbers are denoted $\mathbb{R}$, and the integers $\mathbb{Z}$. The nonnegative reals are $\mathbb{R}_+$, and the nonnegative integers $\mathbb{Z}_+$.

Consider a finite set $E = \{1, \ldots, J\}$. For a vector $x \in \mathbb{R}^J$ and a subset $S \subseteq E$, we write $x(S) = \sum_{j \in S} x_j$, with $x(\emptyset) = 0$.

Consider a nonnegative real-valued function ρ of the subsets of E, $\rho : 2^E \to \mathbb{R}_+$. The set

$$P(\rho) = \left\{ x \in \mathbb{R}_+^J : x(S) \le \rho(S),\ S \subseteq E \right\}$$

is called the polyhedron associated with ρ. Now, suppose ρ satisfies three conditions: ρ is *normalized*, that is, $\rho(\emptyset) = 0$; ρ is *increasing*, that is, $\rho(S) \le \rho(T)$ for $S \subseteq T \subseteq E$; and ρ is *submodular*, that is

$$\rho(S) + \rho(T) \ge \rho(S \cap T) + \rho(S \cup T), \quad S, T \subseteq E.$$

Then, $P(\rho)$ is called a *polymatroid*, and ρ its *rank function*.

Polymatroids were first defined by Edmonds [6], and they have been studied intensively since then. Useful sources include Welsh [35] and Fujishige [13]. In the operations field they have been applied to several areas, including multi-class queues [10,30] and pricing in supply chains [2].

Polymatroids have several remarkable properties. Perhaps the most remarkable is the following: Let p be a vector in $\mathbb{R}_+^J$. Consider the linear program

$$\text{maximize } px, \ \text{ subject to } x \in P(\rho). \tag{1}$$

This problem is easy to solve. Let π be a permutation of E, such that $p_{\pi(1)} \ge p_{\pi(2)} \ge \cdots \ge p_{\pi(J)}$. Define the (unordered) sets $\Pi_j = \{\pi(1), \pi(2), \ldots, \pi(j)\}$, $j = 1, \ldots, J$, and $\Pi_0 = \emptyset$. Define the vector $x^* \in \mathbb{R}_+^J$ by $x_j^* = \rho\left(\Pi_j\right) - \rho\left(\Pi_{j-1}\right)$, $j = 1, \ldots, J$. Then, x^* solves (1). Moreover, if ρ is integer valued ($\rho : 2^E \to \mathbb{Z}_+$), so is x^*. In this case, x^* solves the *integer* linear program (1), with x restricted to $\mathbb{Z}_+^J$. The optimal objective value is

$$px^* = \sum_{j=1}^{J} q_j \rho\left(\Pi_j\right), \tag{2}$$

where $q_j = p_{\pi(j)} - p_{\pi(j+1)} \ge 0$, with $p_{\pi(J+1)} = 0$.

There are several transformations which preserve the polymatroid structure. One of these is called *reduction by a vector*. Let d be any vector in $\mathbb{R}_+^J$. Consider the set

$$P(\rho|d) = \{x \in P(\rho) : x \le d\}.$$

If $P(\rho)$ is a polymatroid, then so is $P(\rho|d)$. Its rank function is denoted $\rho(\cdot|d)$, where

$$\rho(S|d) = \min\{\rho(T) + d(S \setminus T) : T \subseteq S\}. \tag{3}$$

If ρ and d are integer valued, so is $\rho(\cdot|d)$. So, it is also easy to solve linear or integer linear programs of form (1) with additional upper bounds.

Now, suppose $\mathcal{F}$ is a family of subsets of E (that is, a subset of 2^E) including $\emptyset$. Consider a function $\rho : \mathcal{F} \to \mathbb{R}_+$ and the associated polyhedron

$$P(\mathcal{F}, \rho) = \left\{ x \in \mathbb{R}_+^J : x(S) \le \rho(S),\ S \in \mathcal{F} \right\}.$$

(So, $P(\rho) = P(2^E, \rho)$.) We say ρ is increasing, when it is increasing on its domain, that is, $\rho(S) \le \rho(T)$ for $S \subseteq T$, $S, T \in \mathcal{F}$. Likewise, ρ is submodular when it is submodular on its domain. When ρ is normalized, increasing and submodular, call it an $\mathcal{F}$-rank function. In this case, is $P(\mathcal{F}, \rho)$ a polymatroid? That is, does there exist a rank function $\bar{\rho}$ (defined on all of 2^E), such that $P(\mathcal{F}, \rho) = P(\bar{\rho})$? In general, the answer is *no*. But there are some important exceptions, to which we now turn attention.

2.1.2. Sublattices

First, suppose $\mathcal{F}$ is a sublattice of 2^E, that is, closed under $\cap$ and $\cup$. Also, suppose $\mathcal{F}$ contains E as well as $\emptyset$. If ρ is an $\mathcal{F}$-rank function, then indeed $P(\mathcal{F}, \rho)$ is a polymatroid. The function $\bar{\rho}$ is given by $\bar{\rho}(S) = \min\{\rho(T) : S \subseteq T,\ T \in \mathcal{F}\}$, $S \subseteq E$. (It is easy to check that $\bar{\rho}$ is a rank function, and $P(\mathcal{F}, \rho) = P(\bar{\rho})$. This result is due to Edmonds [6]. The formula for $\bar{\rho}$ is due to Federgruen and Groenevelt [8].) So, linear or integer linear programs of form (1), with $P(\mathcal{F}, \rho)$ in place of $P(\rho)$, are again easy to solve.

2.1.3. Tree families

Here is another important special case. A family $\mathcal{F}$ is called a *tree-structured family* (or *tree family* for short), if it can be represented by a binary tree, constructed recursively as follows: Start with the root node, and label it E. At any stage, pick a leaf whose label is a subset having more than one element, say U, and partition it into two subsets, S and T. Add two links to the tree emanating from U, and label the nodes at the other ends S and T. Continue until all the leaves have singleton labels. The family $\mathcal{F}$ comprises the subsets labeling nodes of the tree, plus $\emptyset$. It is easy to see that such a family has exactly $2J$ elements. Note that any such family is *cross-free*. That is, for any pair of subsets in $\mathcal{F}$, either one is contained in the other, or they are disjoint. (A *laminar family* is any subset of a tree family. See [13], page 43. Most of what we say here applies also to laminar families, with suitable adjustments.)

It turns out that, if ρ is a $\mathcal{F}$-rank function, then $P(\mathcal{F}, \rho)$ is a polymatroid. Here,

$$\bar{\rho}(S) = \min\left\{ \begin{array}{l} \sum_{A \in \mathcal{A}} \rho(A) : \mathcal{A} \text{ is a partition of } S \\ \quad \text{with } A \in \mathcal{F}, \ \text{for all } A \in \mathcal{A} \end{array} \right\}. \tag{4}$$

(See [13, page 39].) Such a partition exists, because every singleton $\{j\} \in \mathcal{F}$, $j \in E$. Also, due to the cross-free property, submodularity reduces to a simpler condition:

$$\rho(S) + \rho(T) \ge \rho(U), \quad S, T, U \in \mathcal{F},\ U = S \cup T.$$

That is, ρ is submodular, if and only if it is *subadditive* on $\mathcal{F}$.

We now mention a couple of useful facts. (These are easy to see, but we have not found them in the literature.) First, the minimizing partition $\mathcal{A}$ in (4) depends solely on the structure of $\mathcal{F}$, not on ρ. In fact, it is the *coarsest* partition of S composed of subsets in $\mathcal{F}$. We denote this partition $\mathcal{A}(S)$. It can be found easily as follows: First, find the largest subset in $\mathcal{F}$ contained in S. Call this subset A_1. Second, find the largest subset in $\mathcal{F}$ contained in $S \setminus A_1$, and call it A_2. Continue in this manner until $\{A_i\}$ covers all of S. This partition is clearly minimal in (4), due to the subadditivity of ρ. We record this fact as follows:

Lemma 1.

$$\bar{\rho}(S) = \sum_{A \in \mathcal{A}(S)} \rho(A).$$

Second, consider the reduction of $P(\mathcal{F}, \rho)$ by a vector d, denoted $P(\mathcal{F}, \rho|d)$. One might think that, to apply (3), it is necessary first to extend ρ to $\bar{\rho}$. But there is an easier way. Let us write the tree analogue of (3):

$$\rho(S|d) = \min\{\rho(T) + d(S \setminus T) : T \subseteq S,\ T \in \mathcal{F}\}. \tag{5}$$

It is not hard to check that this calculation correctly accounts for the upper bounds $x \le d$, that is, $P(\mathcal{F}, \rho|d) = P(\mathcal{F}, \rho(\cdot|d))$. Furthermore, $\rho(\cdot|d)$ is a $\mathcal{F}$-rank function. Then we can extend $\rho(\cdot|d)$ to $\bar{\rho}(\cdot|d)$, as above. This is easier than applying (3) directly to $\bar{\rho}$, because (5) involves a much smaller number of subsets. We thus have the following result:

Lemma 2.

$$\bar{\rho}(S|d) = \sum_{A \in \mathcal{A}(S)} \rho(A|d).$$