Operations Research Letters 43 (2015) 226–230

Contents lists available at ScienceDirect

Operations Research Letters

journal homepage: www.elsevier.com/locate/orl

Asymptotic analysis for proximal-type methods in vector variational inequality problems

Zhe Chen *

Business School, Sichuan University, Chengdu 610064, China
Department of Applied Mathematics, The Hong Kong Polytechnic University, Hung Hom Kowloon, Hong Kong

ARTICLE INFO

Article history:
Received 23 October 2014
Received in revised form
5 February 2015
Accepted 5 February 2015
Available online 19 February 2015

Keywords:
Vector variational inequality problems
Proximal-type method
Asymptotic analysis
Weak normal mapping
Convergence analysis

ABSTRACT

In this paper, we present a unified approach for studying vector variational inequality problems in finite dimensional spaces via asymptotic analysis. We introduce a class of weak normal mapping by virtue of the vector-valued indicator function. Then, we employ the obtained results to propose a class of proximal-type method to solve the vector variational inequality problems, carry out convergent analysis on the method and prove convergence of the generated sequence to a solution of the vector variational inequality problems under some mild conditions.

1. Introduction

Let H be a real Hilbert space with inner product $\langle .,.\rangle$, and let $T : H \rightrightarrows H$ be a maximal monotone operator. Consider the following problem: finding an $x \in H$ such that

$$0 \in T(x).$$

This problem is very important in both theory and methodology of mathematical programming and some related fields. One of the efficient algorithms for the above problem is the *proximal point algorithm* (PPA, in short). This algorithm was first introduced by Martinet [18] and its celebrated progress was attained in the work of Rockafellar [20]. The classical proximal point algorithm generated a sequence $\{z^k\} \subset H$ with an initial point z^0 through the following iteration

$$z^{k+1} = (I + c_k T)^{-1} z^k \tag{1.1}$$

where $\{c_k\}$ is a sequence of positive real numbers bounded away from zero. Rockafellar [20] proved that for a maximal monotone operator T, the sequence $\{z^k\}$ weakly converges to a zero of T under some mild conditions. From then on, many works have been devoted to investigate the proximal point algorithm, its applications and generalizations (see [1,16,12,19,24] and the references therein for scalar-valued problems; see [3,4,9,7,6] for vector-valued optimization problems).

On the other hand, the concept of vector variational inequality was firstly introduced by Giannessi [12] in finite dimensional spaces. The vector variational inequality problems have found a lot of important applications in multiobjective decision making problems, network equilibrium problems, traffic equilibrium problems and so on. Because of these significant applications, the study of vector variational inequalities has attracted wide attention. Chen and Yang [10] investigated general vector variational inequality problems and vector complementary problems in infinite dimensional spaces. Chen [5] considered the vector variational inequality problems with a variable ordering structure. Yang [25] studied the inverse vector variational inequality problems and their relations with some vector optimization problems. Recently, Huang, Fang and Yang [15] obtained some necessary and sufficient conditions for the nonemptiness and compactness of the solution set of a pseudomonotone vector variational inequality defined in a finite-dimensional space. Through the last twenty years of development, existence results of solutions, duality theorems and topological properties of solution sets of several kinds of vector variational inequalities have been derived. One can find a fairly complete review of the main results about vector variational inequalities in the monograph [8] and in the survey paper [13].

However to the best of our knowledge, there is no numerical method has been designed for solving vector variational inequality problems, even no conceptual one. Motivated by the classical

* Correspondence to: Business School, Sichuan University, Chengdu 610064, China.
E-mail address: mazhechen@gmail.com.

http://dx.doi.org/10.1016/j.orl.2015.02.003
0167-6377/

results [20] of Rockafellar's, in this paper we firstly try to construct a class of vector-valued proximal-type method for solving a weak vector variational inequality problem and prove the sequence generated by our method converges to a solution of the weak vector variational inequality problem under some mild conditions.

The paper is organized as follows.

In Section 2, we present some basic concepts, assumptions and preliminary results. In Section 3, we introduce the proximal-type method and carry out convergence analysis on the method.

2. Preliminaries

In this section, we present some basic definitions and propositions for the proof of our main results.

Let $C = R^m_+ \subset R^m$ and $C_1 = \{x \in R^m_+ | \, \|x\| = 1\}$. We define, for any $y_1, y_2 \in R^m$,

$$y_1 \leq_C y_2 \quad \text{if and only if } y_2 - y_1 \in C;$$
$$y_1 \not\leq_{\text{int}C} y_2 \quad \text{if and only if } y_2 - y_1 \notin \text{int}C.$$

The extended space of R^m is $\bar{R}^m = R^m \cup \{-\infty_C, +\infty_C\}$, where $-\infty_C$ is an imaginary point, each of the coordinates is $-\infty$ and the imaginary point $+\infty_C$ is analogously understood (with the conventions $\infty_C + \infty_C = \infty_C, \mu(+\infty_C) = +\infty_C$ for each positive number μ). The point $y \in R^m$ is a column vector and its transpose is denoted by $y^\top$. The inner product in R^m is denoted by $\langle \cdot, \cdot \rangle$.

Let X_0 be a nonempty subset of R^n and let $T_i : X_0 \to R^n$, $i \in [1, \ldots, m]$ be vector-valued functions. Let $T := (T_1, \ldots, T_m)$ be a $n \times m$ matrix which columns are $T_i(x)$, and let

$$T(x) = (T_1(x), \ldots, T_m(x)),$$
$$T(x)^\top(v) = (\langle T_1(x), v \rangle, \ldots, \langle T_m(x), v \rangle)^\top$$

for every $x \in X_0$ and $v \in R^n$. For any $\lambda \in C_1$, a mapping $\lambda(T) : X_0 \to R^n$ is defined by

$$\lambda(T)(x) = \Sigma_{i=1}^m \lambda_i T_i(x), \quad x \in X_0. \tag{2.1}$$

Definition 2.1 (*[12]*). A vector variational inequality (VVI in short) is a problem of finding $x^* \in X_0$ such that

$$\text{(VVI)}\ T(x^*)^\top(x - x^*) \not\leq_{C \setminus \{0\}} 0, \quad \forall x \in X_0,$$

where x^* is called a solution of problem (VVI).

Definition 2.2 (*[8]*). A weak variational inequality (WVVI in short) is a problem of finding $x^* \in X_0$ such that

$$(WVVI)\ T(x^*)^\top(x - x^*) \not\leq_{\text{int}C} 0, \quad \forall x \in X_0,$$

where x^* is called a solution of problem (WVVI). Denote by X^* the solution set of problem (WVVI).

Let $\lambda \in C_1$, consider the corresponding scalar-valued variational inequality problem of finding $x^* \in X_0$ such that:

$$(\text{VIP}_\lambda)\ \langle \lambda(T)(x^*), x - x^* \rangle \geq 0, \quad \forall x \in X_0.$$

Denote by X^*_λ be the solution set of (VIP_λ).

It is worth noticing that the partial order $\not\leq_{\text{int}C}$ is closed in the sense that if $x_k \to x^*$ as $k \to \infty$, $x_k \not\leq_{\text{int}C} 0$, then we have $x^* \not\leq_{\text{int}C} 0$. This is because of the closeness of the set $S =: R^m \setminus (-\text{int}C)$.

Definition 2.3 (*[15]*). Let $X_0 \subset R^n$ be nonempty, closed and convex, and $F : X_0 \to R^n$ be a single-valued mapping.

(i) F is said to be *monotone* on X_0 if, for any $x_1, x_2 \in X_0$, there holds

$$\langle F(x_1) - F(x_2), x_1 - x_2 \rangle \geq 0.$$

(ii) F is said to be *pseudomonotone* X_0 if, for any $x_1, x_2 \in X_0$, there holds

$$\langle F(x_2), x_1 - x_2 \rangle \geq 0 \Rightarrow \langle F(x_1), x_1 - x_2 \rangle \geq 0.$$

Clearly, a monotone map is pseudomonotone.

Now we give the definitions of C-monotonicity of a matrix-valued map.

Definition 2.4 (*[8]*). Let $X_0 \subset R^n$ be nonempty, closed and convex. $T : X_0 \to R^{n \times m}$ is a mapping, which is said to be C-*monotone* on X_0 if, for any $x_1, x_2 \in X_0$, there holds

$$(T(x_1) - T(x_2))^\top(x_1 - x_2) \geq_C 0.$$

Proposition 2.1 (*[15]*). *Let X_0 and T be defined as in Definition 2.4, we have the following statements:*

(i) *T is C-monotone if and only if, for any $\lambda \in C_1$, the mapping $\lambda(T) : X_0 \to R^n$ defined by (2.1) is monotone.*
(ii) *if T is C-monotone, then for any $\lambda \in C_1$, $\lambda(T) : X_0 \to R^n$ is pseudomonotone.*

Definition 2.5 (*[14]*). Let $L \subset R^{n \times m}$ be a nonempty set. The weak and strong C-polar cones of L are defined, respectively, by

$$L^{w_0}_C := \{x \in R^n : l(x) \not\geq_C 0, \ \forall l \in L\}; \tag{2.2}$$

and

$$L^{s_0}_C := \{x \in R^n : l(x) \leq_C 0, \ \forall l \in L\}. \tag{2.3}$$

Definition 2.6 (*[8]*). Let $K \subset R^n$ be nonempty, closed and convex, $F : K \subset R^n \to R^m \cup \{+\infty_C\}$ be a vector-valued mapping. A $n \times m$ matrix V is said to be a strong subgradient of F at $\bar{x} \in K$ if

$$F(x) - F(\bar{x}) - V^\top(x - \bar{x}) \geq_C 0 \quad \forall x \in K.$$

A $n \times m$ matrix V is said to be a weak subgradient of F at $\bar{x} \in K$ if

$$F(x) - F(\bar{x}) - V^\top(x - \bar{x}) \not\leq_{\text{int}C} 0 \quad \forall x \in K.$$

Denote by $\partial^w_C F(\bar{x})$ the set of weak subgradients of F on K at $\bar{x}$.

Let $K \subset R^n$ be nonempty, closed and convex. A vector-valued indicator function $\delta(x \mid K)$ of K at x is defined by

$$\delta(x \mid K) = \begin{cases} 0 \in R^n, & \text{if } x \in K; \\ +\infty_C, & \text{if } x \notin K. \end{cases}$$

An important and special case in the theory of weak subgradient is that when $F(x) = \delta(x \mid K)$ becomes a vector-valued indicator function of K. By Definition 2.6, we obtain $V \in \partial^w_C \delta(x^* \mid K)$ if and only if

$$V^\top(x - x^*) \not\geq_{\text{int}C} 0 \quad \forall x \in K. \tag{2.4}$$

Definition 2.7. A set $VN^w_K(x^*) \subset R^{n \times m}$ is said to be a weak normality operator set to K at x^*, if for every $V \in VN^w_K(x^*)$, the inequality (2.4) holds.

Clearly, $VN^w_K(x^*) = \partial^w_C \delta(x^* \mid K)$. As for the scalar-valued case, from [21] we know that $v^* \in \partial \delta_K(x^*) = N_K(x^*)$ if and only if

$$\langle v^*, x - x^* \rangle \leq 0 \quad \forall x \in K \tag{2.5}$$

where $\delta_K(x)$ is the scalar-valued indicator function of K. The inequality (2.5) means that v^* is normal to K at x^*.

Definition 2.8. Let $VN^w_K(.) : R^n \rightrightarrows R^{n \times m}$ be a set-valued mapping, which is said to be a weak normal mapping for K, if for any $y \in K$, $V \in VN^w_K(y)$ such that

$$V^\top(x - y) \not\geq_{\text{int}C} 0, \quad \forall x \in K. \tag{2.6}$$

$VN^s_K(.)$ is said to be strong normal mapping for K, if for any $y \in K$, $V \in VN^s_K(y)$ such that

$$V^\top(x - y) \leq_C 0, \quad \forall x \in K. \tag{2.7}$$