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# Inverse optimization in countably infinite linear programs

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ABSTRACT

Given the costs and a feasible solution for a linear program, inverse optimization involves finding new costs that are close to the original ones and make the given solution optimal. We develop an inverse optimization framework for countably infinite linear programs using the weighted absolute sum metric. We reformulate this as an infinite-dimensional mathematical program using duality. We propose a convergent algorithm that solves a sequence of finite-dimensional LPs to tackle it. We apply this to non-stationary Markov decision processes.



## 1. Introduction

Countably infinite linear programs (CILPs) are infinite-dimensional linear programs (LPs) that include a countable number of variables and a countable number of constraints [3,9,10]. CILPs arise in infinite-horizon planning applications such as production planning, equipment replacement, and capacity expansion. Special cases of CILPs include minimum cost flow problems on infinite networks [16,20]; infinite horizon stochastic programs [11]; LP formulations of countable-state Markov decision processes (MDPs) [10,13,15,19]; and problems in robust optimization [8].

Inverse optimization in $n$-dimensional LPs refers to the following problem: given a feasible solution $x^* \in \Re^n$ to an LP with cost vector $c^* \in \Re^n$, find a $d \in \Re^n$ that (i) is as close (in an appropriate distance metric) as possible to $c^*$, and (ii) makes $x^*$ optimal to a new LP where the cost vector is $d$. Ahuja and Orlin [1] showed that if we used the weighted absolute sum metric in $\Re^n$, then this problem reduces to a finite-dimensional LP. Chan et al. [5] stated that inverse optimization has been studied in the context of shortest path problems; network and combinatorial optimization; integer programming; mixed integer programming; and convex optimization. Despite the recent surge of interest in CILPs, inverse optimization has not yet been studied in the CILP context. The goal of this paper is to develop an inverse optimization framework for CILPs.

We pursue a duality-based as in Ahuja and Orlin. However, the difficulty is that unlike finite-dimensional LPs, weak duality, complementary slackness, and strong duality may not hold in general in primal–dual pairs of CILPs [3,8,14,17,18]. It is essential to embed the primal and the dual CILPs in appropriately chosen sequence spaces to ensure that weak duality and complementary slackness hold, and then to impose additional restrictions for strong duality to hold. This task is rendered difficult owing to mathematical pathologies in sequence spaces and has been called the "Slater conundrum" [14]. The duality approach in Ghate [8] allows for the broadest class of CILPs. That framework is therefore utilized to cast CILP formulations here.

Following Ahuja and Orlin, we use the weighted absolute sum metric. The weights are embedded in an appropriate sequence space so that the corresponding metric is finite. We show that the constraints in our inverse optimization problem can be equivalently reformulated as a countably infinite set of linear constraints. We propose to solve a sequence of finite-dimensional LPs to tackle the resulting infinite-dimensional inverse optimization problem. We prove that accumulation points of any sequence of optimal solutions to these finite-dimensional LPs are optimal to the inverse optimization problem. We also prove that the sequence of optimal values of these finite-dimensional LPs converges to the optimal value of the inverse optimization problem. These results are applied to infinite-horizon *non-stationary* MDPs, thus extending recent work on inverse optimization in infinite-horizon *stationary* MDPs [7]. Proofs are provided in the supplementary material available via author's website at http://faculty.washington.edu/archis/orl-inverse-opt-suppl.pdf.

## 2. Review of duality in CILPs

We first review CILP duality results from Ghate [8]. The symbol $\triangleq$ means "defined as". We use $\mathbb{N} \triangleq \{1, 2, \ldots\}$ to denote the set of all natural numbers and let $\Re^{\mathbb{N}}$ denote the space of all real-valued

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sequences. Let $Z \subseteq \Re^{\mathbb{N}}$ be a sequence space. Generic sequences in $Z$ will often be denoted by $c$ and will form the objective function coefficients in our CILPs. Let $b \in \Re^{\mathbb{N}}$ be a given sequence; and, for $i = 1, 2, \ldots$, let $A_{i\cdot} \triangleq (a_{i1}, a_{i2}, \ldots) \in \Re^{\mathbb{N}}$ be the $i$th row of a given doubly-infinite matrix $A$. Similarly, let $A_{\cdot j} \triangleq (a_{1j}, a_{2j}, \ldots) \in \Re^{\mathbb{N}}$ be the $j$th column of this matrix. As is common in the existing literature, we assume that, for each $i$, only a finite number of entries in $A_{i\cdot}$ is non-zero; similarly, we assume that, for each $j$, only a finite number of entries in $A_{\cdot j}$ is non-zero. This structure is ubiquitous (see [17,18]) in Operations Research such as in shortest path formulations of infinite-horizon dynamic programs with finite states and actions [9]; minimum cost flow problems in infinite-staged networks with finite node degrees [16,20]; and CILP formulations of infinite-horizon non-stationary MDPs with finite states and actions [10,12].

Now let $X \subseteq \Re^{\mathbb{N}}$ be the subspace of all sequences $x \in \Re^{\mathbb{N}}$ for which

C1. the series $C(x) \triangleq \sum_{j=1}^{\infty} c_j x_j$ converges for any $c \in Z$.

Let $Y$ be the subspace of all $y \in \Re^{\mathbb{N}}$ for which

C2. the series $B(y) \triangleq \sum_{i=1}^{\infty} b_i y_i$ converges; and

C3. for every $x \in X$, we have $\sum_{i=1}^{\infty} L_i(x, y_i) < \infty$, where $L_i(x, y_i) \triangleq \sum_{j=1}^{\infty} |a_{ij} x_j y_i|$ for each $i = 1, 2, \ldots$.

Consider the following pair of primal–dual CILPs

$$(P)\ V(P) = \inf \sum_{j=1}^{\infty} c_j x_j \tag{1}$$

$$\sum_{j=1}^{\infty} a_{ij} x_j = b_i, \quad i = 1, 2, \ldots, \tag{2}$$

$$x_j \geq 0, \quad j = 1, 2, \ldots, \tag{3}$$

$$x \in X, \tag{4}$$

and

$$(D)\ V(D) = \sup \sum_{i=1}^{\infty} b_i y_i \tag{5}$$

$$\sum_{i=1}^{\infty} a_{ij} y_i \leq c_j, \quad j = 1, 2, \ldots, \tag{6}$$

$$y \in Y. \tag{7}$$

Let $F$ and $G$ denote the feasible regions of these two problem, respectively.

**Theorem 2.1** (*Weak Duality: Ghate [8]*). *For any $x \in F$ and any $y \in G$, $\sum_{j=1}^{\infty} c_j x_j \geq \sum_{i=1}^{\infty} b_i y_i$. Hence $\infty \geq V(P) \geq V(D) \geq -\infty$ (here, the infimum over an empty set is interpreted as $+\infty$ and the supremum over an empty set is interpreted as $-\infty$). Also, if $x \in F$ and $y \in G$ are such that $\sum_{j=1}^{\infty} c_j x_j = \sum_{i=1}^{\infty} b_i y_i$, then $x$ is optimal to $(P)$ and $y$ is optimal to $(D)$, and thus strong duality holds.*

Vectors $x \in X$ and $y \in Y$ are called complementary if $x_j\big(c_j - \sum_{i=1}^{\infty} a_{ij} y_i\big) = 0$ for each $j = 1, 2, \ldots$.

**Theorem 2.2** (*Complementary Slackness: Ghate [8]*).

1. *Suppose $x \in F$ and $y \in G$, and suppose $x$ and $y$ are complementary. Then $x$ is optimal to $(P)$, $y$ is optimal to $(D)$, and $V(P) = V(D)$. Thus, strong duality holds in this case.*
2. *Suppose $x$ is optimal to $(P)$, $y$ is optimal to $(D)$, and $V(P) = V(D)$ (that is, strong duality holds). Then $x$ and $y$ are complementary.*

For any increasing sequences of positive integers $N_n$ and $M_n$, consider the truncation given by

$$P(n)\ V(P(n)) = \inf \sum_{j=1}^{M_n} c_j x_j \tag{8}$$

$$\sum_{j=1}^{M_n} a_{ij} x_j = b_i, \quad i = 1, 2, \ldots, N_n, \tag{9}$$

$$x_j \geq 0, \quad j = 1, 2, \ldots, M_n. \tag{10}$$

Let $X^*(n) \subseteq X$ denote the (possibly empty) set of optimal solutions to $P(n)$. The dual of $P(n)$ is

$$D(n)\ V(D(n)) = \sup \sum_{i=1}^{N_n} b_i y_i \tag{11}$$

$$\sum_{i=1}^{N_n} a_{ij} y_i \leq c_j, \quad j = 1, 2, \ldots, M_n. \tag{12}$$

Let $Y^*(n) \subseteq Y$ denote the (possibly empty) set of optimal solutions to $D(n)$.

We use the product topology on sequence spaces in $\Re^{\mathbb{N}}$.

**Theorem 2.3** (*Strong Duality: Ghate [8]*). *Suppose there exist the aforementioned sequences $P(n)$ and $D(n)$ of finite-dimensional primal–dual problems and sets $\mathcal{C} \subseteq X$ and $\mathcal{K} \subseteq Y$ such that*

C4. *for each $n$, $X_{\mathcal{C}}(n) \triangleq \left(X^*(n) \bigcap \mathcal{C}\right) \neq \emptyset$;*

C5. *for each $n$, $Y_{\mathcal{K}}(n) \triangleq \left(Y^*(n) \bigcap \mathcal{K}\right) \neq \emptyset$;*

C6. *$\exists$ a sequence in $X_{\mathcal{C}}(n) \times Y_{\mathcal{K}}(n)$ with a convergent subsequence with a limit in $\mathcal{C} \times \mathcal{K}$.*

*Then $(P)$ and $(D)$ have optimal solutions in $\mathcal{C}$ and $\mathcal{K}$, and $V(P) = V(D)$.*

The product topologies on $X$ and $Y$ are countable products of the usual metrizable topology on $\Re$; hence they are metrizable (see Theorem 3.36 on p. 89 of [2]). Every sequence in a compact set in the product topology on $X$ thus has a convergent subsequence; similarly for $Y$ (see Theorem 3.28 on p. 86 of [2]). Condition C6 thus holds when $\mathcal{C}$ and $\mathcal{K}$ are compact. By the Tychonoff product theorem (see Theorem 2.61 on p. 52 of [2]), this compactness holds if each component of optimal solutions to $P(n)$ and each component of optimal solutions to $D(n)$ can be bounded independently of $n$. See [8,17,18] for several applications where these conditions are met.

## 3. Inverse optimization formulation

Now suppose that a fixed $c^* \in Z$ is given. Also suppose that an $x^* \in X$ that is feasible to $(P)$ is given. For each $d \in Z$, we will use $(P_d)$, for emphasis, to denote the CILP that is identical in form to $(P)$ but now with cost coefficients $d$. We will use $(D_d)$ to denote the dual of $(P_d)$. Note that $(P_d)$ and $(D_d)$ satisfy C1–C3 and hence weak duality as in Theorem 2.1 and complementary slackness as in Theorem 2.2 hold for this primal–dual pair.

Following Ahuja and Orlin, we say that $d \in Z$ is *inverse feasible* with respect to $x^*$ if $x^*$ is an optimal solution to $(P_d)$. Similarly, we will use $\mathcal{D}(x^*) \subseteq Z$ to denote the set of all $d \in Z$ that are inverse feasible with respect to $x^*$. Our inverse optimization problem involves finding a $d \in \mathcal{D}(x^*) \cap \mathcal{C}$ that is as close as possible to $c$ in the weighted absolute sum metric, where $\mathcal{C} \triangleq \{d \in Z : l_j \leq d_j \leq u_j,\ \forall j\}$ is some compact set in the product topology on $Z$. Here, the given lower and upper bound vectors $l, u$ belong to $Z$. Let