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Robust counterparts and robust efficient solutions in vector optimization under uncertainty

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ABSTRACT

Two robust counterparts and associated concepts of robust efficient solution are established for a vector optimization problem under uncertainty. First, we propose a robust counterpart in the classical sense by following the line for scalar optimization problems under uncertainty. Then, from a relaxed model we derive another robust counterpart, which is a bilevel optimization problem involving a set-valued optimization problem at the upper level and a vector optimization problem at the lower level.

1. Introduction and preliminaries

In the last two decades, robust optimization has been through a rapid development owing to the practical requirement and its effective implementation in real-world applications of optimization [2,3,8]. Recently much attention has been paid to introducing robustness in multi-objective optimization under uncertainty. For a multi-objective optimization problem with the objective function involving an uncertain parameter, several concepts of robustness were developed without scalarization, such as the component-wise worst case robustness in [17,7], min–max robustness in [5], convex version of min–max robustness in [4] and other ones based on set order relations in [12,13,11]. In this study, we consider the following vector optimization problem

$$\operatorname*{Min}_{x\in D} f_u(x) := f(x,u) \tag{1.1}$$

where u is an uncertain parameter varying in a bounded closed set $U \subset R^N$ and $f_u : D \subset R^n \to R^m$ is a vector-valued function.

Let X, Y and Z be three topological linear spaces and K be a proper, pointed and convex cone in Y. Then Y is partially ordered by $\leq_K$, i.e.

$$y_1 \leq_K y_2 \Leftrightarrow y_2 - y_1 \in K.$$

An associated order $\leq_{K\setminus\{0\}}$ is defined by

$$y_1 \leq_{K\setminus\{0\}} y_2 \Leftrightarrow y_2 - y_1 \in K \setminus \{0\}.$$

Meanwhile, set order relations in the power space 2^Y can be constructed on the basis of Q ($= K$ or $K \setminus \{0\}$). For two sets B and C in Y, we write that

$$B \preccurlyeq_Q^l C \quad \text{if } C \subset B + Q$$

and

$$B \preccurlyeq_Q^u C \quad \text{if } B \subset C - Q.$$

Following Jahn and Ha [15], we call $\preccurlyeq_Q^l$ the l-type less order relation, $\preccurlyeq_Q^u$ the u-type less order relation.

For a set $A \subset Y$, recall that

(i) $\bar{a}$ is a minimal element of A if $\bar{a} \in A$ and there is no $a \in A \setminus \{\bar{a}\}$ such that $a \leq_K \bar{a}$;
(ii) $\bar{a}$ is a maximal element of A if $\bar{a} \in A$ and there is no $a \in A \setminus \{\bar{a}\}$ such that $\bar{a} \leq_K a$;
(iii) $\tilde{a}$ is a lower bound of A if $\tilde{a} \leq_K a$ for all $a \in A$;
(iv) $\tilde{a}$ is an upper bound of A if $a \leq_K \tilde{a}$ for all $a \in A$;
(v) $\hat{a}$ is an infimum of A if $\hat{a}$ is a lower bound and $\tilde{a} \leq_K \hat{a}$ for every lower bound $\tilde{a}$;
(vi) $\hat{a}$ is a supremum of A if $\hat{a}$ is an upper bound and $\hat{a} \leq_K \tilde{a}$ for every upper bound $\tilde{a}$.

The minimal set, maximal set, infimum and supremum of A are denoted by $\operatorname{Min} A$, $\operatorname{Max} A$, $\inf A$ and $\sup A$, respectively. It is worth noting that for a set in a partially ordered space the infimum is unique if it exists, so is the supremum [18].

Suppose that $f : D \times U \to Y$ is a vector-valued function with $D \subset X$ and $U \subset Z$. For problem (1.1) with $u \in U$ fixed,

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(1) $\bar{x} \in D$ is an efficient solution if there is no $x \in D$ such that $f_u(x) \leq_{K\setminus\{0\}} f_u(\bar{x})$;
(2) $\bar{x} \in D$ is a strictly efficient solution if there is no $x \in D \setminus \{\bar{x}\}$ such that $f_u(x) \leq_K f_u(\bar{x})$.

It is well known that

(3) $\bar{x}$ is an efficient solution if and only if $f_u(\bar{x}) \in \mathrm{Min}\, f_u(D)$;
(4) $\bar{x}$ is a strictly efficient solution if and only if $f_u(\bar{x}) \in \mathrm{Min}\, f_u(D)$ and $f_u(\bar{x}) \notin f_u(D \setminus \{\bar{x}\})$,

where $f_u(D) := \{f_u(x) : x \in D\}$. Additionally, throughout this paper we denote $F(x) = \{f(x,u) : u \in U\}$ and assume that $\mathrm{Max}\, F(x)$ is nonempty and $\sup F(x)$ exists for every $x \in D$.

2. Robust counterparts and robust efficient solutions

In scalar optimization, the original robustness in the hard sense means immunity against all scenarios. This leads to the worst case model, which is a min–max or max–min problem, in the case of the scalar optimization problem under objective-wise uncertainty. For problem (1.1) with $Y = R$ the real space and $K = R_+$ the routine cone, its robust counterpart is

$$\min_{x\in D} \max_{u\in U} f(x,u) \tag{2.1}$$

under the assumption that there exists a $\bar{u} \in U$ such that $f(x,\bar{u}) = \sup_{u\in U} f(x,u)$. This model can be derived in two steps: (i) write problem (1.1) as

$$\begin{aligned} \min_{x,t} \quad & t \\ \text{s.t.} \quad & f(x,u) \leq t \\ & x \in D, \end{aligned}$$

(ii) robustify the problem above with respect to u as

$$\begin{aligned} \min_{x,t} \quad & t \\ \text{s.t.} \quad & f(x,u) \leq t, \quad \forall u \in U \\ & x \in D, \end{aligned}$$

that is,

$$\begin{aligned} \min_{x,t} \quad & t \\ \text{s.t.} \quad & \sup_{u\in U} f(x,u) \leq t \\ & x \in D, \end{aligned}$$

which is problem (2.1).

We follow the same line to deduce a robust counterpart for problem (1.1) in the vector optimization setting. We begin with a lemma about transforming the vector-valued objective function into a vector inequality constraint.

Lemma 2.1. *Let X and Y be two topological linear spaces with Y partially ordered by $\leq_K$. Suppose $g : X \to Y$. Then for the following two problems*

$$\mathrm{Min}_{x\in X}\ g(x) \tag{2.2}$$

and

$$\begin{aligned} \mathrm{Min}_{x,t} \quad & t \\ \text{s.t.} \quad & g(x) \leq_K t \\ & x \in X, \end{aligned} \tag{2.3}$$

we have

(a) *$\bar{x}$ is an efficient solution to problem (2.2) if and only if $(\bar{x}, \bar{t})$ is an efficient solution to problem (2.3) for some $\bar{t}$;*
(b) *$\bar{x}$ is a strictly efficient solution to problem (2.2) if and only if $(\bar{x}, \bar{t})$ is a strictly efficient solution to problem (2.3) for some $\bar{t}$.*

Proof. (a) The cone K is pointed and convex, then

$$\begin{aligned} \mathrm{Min}\, g(X) &= \mathrm{Min}\,(g(X) + K) \\ &= \mathrm{Min}\{t : g(x) \leq_K t, x \in X\}. \end{aligned} \tag{2.4}$$

The necessity is verified immediately by the equalities (2.4) with $\bar{t} = g(\bar{x})$. The proof of the sufficiency begins with showing $\bar{t} = g(\bar{x})$ whenever $(\bar{x}, \bar{t})$ is an efficient solution to problem (2.3). In fact, if $\bar{t} \neq g(\bar{x})$, then $g(\bar{x}) \leq_{K\setminus\{0\}} \bar{t}$, which contradicts $\bar{t} \in \mathrm{Min}\{t : g(x) \leq_K t, x \in X\}$. Then the equalities (2.4) leads to $g(\bar{x}) \in \mathrm{Min}\, g(X)$, that is, $\bar{x}$ is an efficient solution to problem (2.2).

(b) Suppose that $\bar{x}$ is a strictly efficient solution to problem (2.2). Let $\bar{t} = g(\bar{x})$. Then there is no $(x,t) \neq (\bar{x}, \bar{t})$ with $x \neq \bar{x}$ and $g(x) \leq_K t$ such that $t \leq_K \bar{t}$. Otherwise, the transitivity of $\leq_K$ results in that there is another $x \in X$ such that $g(x) \leq_K g(\bar{x})$, which contradicts the strict efficiency of $\bar{x}$. On the other hand, for every feasible point $(\bar{x}, t) \neq (\bar{x}, \bar{t})$ with $t \neq \bar{t}$, it is impossible that $t \leq_K \bar{t}$ since $g(\bar{x}) \leq_K t$. In all, there is no $(x,t) \neq (\bar{x}, \bar{t})$ with $g(x) \leq_K t$ such that $t \leq_K \bar{t}$, that is, $(\bar{x}, \bar{t})$ is a strictly efficient solution to problem (2.3).

The reverse can be discussed analogously with a slight difference. We need to prove $\bar{t} = g(\bar{x})$ first whenever $(\bar{x}, \bar{t})$ is a strictly efficient solution to problem (2.3). In fact, if not, there exists a point $(\bar{x}, g(\bar{x})) \neq (\bar{x}, \bar{t})$ such that $g(\bar{x}) \leq_K \bar{t}$. Next, consider every $x \neq \bar{x}$. Then $g(x) \leq_K \bar{t}$ cannot hold since $(x, g(x)) \neq (\bar{x}, \bar{t})$. This means that there is no $x \in X \setminus \{\bar{x}\}$ such that $g(x) \leq_K g(\bar{x})$, that is, $\bar{x}$ is a strictly efficient solution to problem (2.2). □

Remark 2.1. Note that $\mathrm{Min}\, g(X)$ may not equal $\mathrm{Min}\,(g(X) + K)$ if K is not pointed, while $\mathrm{Min}\, g(X) \subset \mathrm{Min}\,(g(X) + K)$ holds always provided that K is a convex cone [14]. For the case where K lacks pointedness, problem (2.2) may not be equivalent to problem (2.3) but can be deemed a safe approximation of problem (2.3).

Now set $g(\cdot) := f_u(\cdot)$, then for problem (1.1) we get

$$\begin{aligned} \mathrm{Min}_{x,t} \quad & t \\ \text{s.t.} \quad & f(x,u) \leq_K t \\ & x \in D. \end{aligned}$$

After introducing robustness with respect to u, we have

$$\begin{aligned} \mathrm{Min}_{x,t} \quad & t \\ \text{s.t.} \quad & f(x,u) \leq_K t, \quad \forall u \in U \\ & x \in D, \end{aligned} \tag{2.5}$$

which is

$$\begin{aligned} \mathrm{Min}_{x,t} \quad & t \\ \text{s.t.} \quad & \sup F(x) \leq_K t \\ & x \in D. \end{aligned} \tag{2.6}$$

Since $\sup F(\cdot)$ is single-valued, then by Lemma 2.1 again the last problem is equivalent to

$$\mathrm{Min}_{x\in D}\ \sup F(x). \tag{2.7}$$

Definition 2.1. Problem (2.7) is called the robust counterpart in the hard sense or H-robust counterpart of problem (1.1). Every efficient solution to problem (2.7) is called the H-robust efficient solution to problem (1.1); every strictly efficient solution to problem (2.7) is called the H-robust strictly efficient solution to problem (1.1).

Remark 2.2. It is obvious that robust counterpart (2.7) reduces to robust counterpart (2.1) when problem (1.1) is a scalar optimization problem in that $\sup F(x) = \max_{u\in U} f(x,u)$.