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Some lower bounds on sparse outer approximations of polytopes

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ABSTRACT

As a means to understand the use of sparse cutting-planes in integer programming solvers, the paper Dey et al. (2014) studied how well polytopes are approximated by using only sparse valid-inequalities. We consider "less-idealized" questions such as: effect of sparse inequalities added to linear-programming relaxation, effect on approximation by addition of a budgeted number of dense valid-inequalities, sparse-approximation of polytope under every rotation and approximation by sparse inequalities in specific directions.

1. Introduction

The paper [2] studied how well one can expect to approximate polytopes using valid inequalities that are sparse. The motivation for this study came from the usage of cutting-planes in integer programming (IP) solvers. In principle, facet-defining inequalities of the integer hull of a polytope can be dense, i.e. they can have non-zero coefficients for a high number of variables. In practice, however, most state-of-the-art IP solvers bias their cutting-plane selection towards the use of sparse inequalities. This is done, in part, to take advantage of the fact that linear programming solvers can harness sparsity well to obtain significant speedups.

The paper [2] shows that for polytopes with a polynomial number of vertices, sparse inequalities produce very good approximations of polytopes. However, when the number of vertices increase, the sparse inequalities do not provide a good approximation in general; in fact with high probability the quality of approximation is poor for random 0–1 polytopes with super polynomial number of vertices (see details in [2]).

However the study in [2] is very "idealized" in the context of cutting-planes for IPs, since almost always some dense cutting-planes are used or one is interested in approximating the integer only along certain directions. In this paper, we consider some natural extensions to understand the properties of sparse inequalities under more "realistic conditions":

1. All the results in the paper [2] deal with the case when we are attempting to approximate the integer hull using only sparse inequalities. However, in practice the LP relaxation may have dense inequalities. Therefore we examine the following question: Are there integer programs, such that sparse inequalities do not approximate the integer hull well when added to a linear programming relaxation?
2. More generally, we may consider attempting to improve the approximation of a polytope by adding a few dense inequalities together with sparse inequalities. Therefore we examine the following question: Are there polytopes, where the quality of approximation by sparse inequalities cannot be significantly improved by adding polynomial (or even exponential) number of *arbitrary* valid inequalities?
3. It is clear that the approximations of polytopes using sparse inequalities is not invariant under affine transformations (in particular rotations). This leaves open the possibility that a clever reformulation of the polytope of interest may vastly improve the approximation obtained by sparse cuts. Therefore a basic question in this direction: Are there polytopes that are difficult to approximate under *every* rotation?
4. In optimization one is usually concerned with the feasible region in the direction of the objective function. Therefore we examine the following question: Are there polytopes that are difficult to approximate in *almost all* directions using sparse inequalities?

We are able to present examples that answer each of the above questions in the positive. This is perhaps not surprising: an indication that sparse inequalities do not always approximate

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integer hulls well even in the more realistic settings considered in this paper. Understanding when sparse inequalities are effective in all the above settings is an important research direction.

The rest of the paper is organized as follows. Section 2 collects all required preliminary definitions. In Section 3 we formally present all the results. In Sections 4–7 we present proofs of the various results.

2. Preliminaries

2.1. Definitions

For a natural number n, let $[n]$ denote the set $\{1, \ldots, n\}$ and, for non-negative integer $k \leq n$ let $\binom{[n]}{k}$ denote the set of all subsets of $[n]$ with k elements. For any $x \in \mathbb{R}^n$, let $\|x\|_1$ denote the l_1 norm of x and $\|x\|$ or $\|x\|_2$ denote the l_2 norm of x.

An inequality $\alpha x \leq \beta$ is called *k-sparse* if α has at most k non-zero components. Given a polytope $P \subset \mathbb{R}^n$, P^k is defined as the intersection of all k-sparse cuts valid for P (as in [2]), that is, the best outer-approximation obtained from k-sparse inequalities. We remark that P^k is also a polytope (see [2]).

Given two polytopes $P, Q \subset \mathbb{R}^n$ such that $P \subseteq Q$ we consider the Hausdorff distance $d(P, Q)$ between them:

$$d(P, Q) := \max_{x \in Q} \left(\min_{y \in P} \|x - y\| \right).$$

When $P, Q \subset [-1, 1]^n$, we have that $d(P, Q)$ is upper bounded by $2\sqrt{n}$, the largest distance between two points in $[-1, 1]^n$. In this case, if $d(P, Q) \propto \sqrt{n}$ the error of approximation of P by Q is basically as large as it can be and smaller $d(P, Q)$ (for example constant or of the order of $\sqrt{\log n}$) will indicate better approximations.

Given a polytope $P \subseteq \mathbb{R}^n$ and a vector $c \in \mathbb{R}^n$, we define

$$gap_P^k(c) = \max_{x \in P^k} cx - \max_{x \in P} cx,$$

namely the "gap" between P^k and P in direction c. We first note that $d(P, P^k)$ equals the worst directional gap between P^k and P (the proof is presented in Appendix A).

Lemma 1. *For every polytope* $P \subseteq \mathbb{R}^n$, $d(P, P^k) = \max_{c:\|c\|=1} gap_P^k(c)$.

For a set $\mathcal{D} = \{\alpha_1 x \leq \beta_1, \ldots, \alpha_d x \leq \beta_d\}$ of (possibly dense) valid inequalities for P, let $P^{k,\mathcal{D}}$ denote the outer-approximation obtained by adding all k-sparse cuts and the inequalities from $\mathcal{D}$:

$$P^{k,\mathcal{D}} = \left(\bigcap_{i=1}^{d} \left\{ x \in \mathbb{R}^n : a_i x \leq b_i \right\} \right) \bigcap P^k. \tag{1}$$

Since $P^{k,\mathcal{D}} \subseteq P^k$ we have that $d(P, P^{k,\mathcal{D}}) \leq d(P, P^k)$ for any set $\mathcal{D}$ of valid inequalities for P.

2.2. Important Polytopes

Throughout the paper, we will focus our attention on the polytopes $\mathcal{P}_{t,n} \subseteq [0, 1]^n$ defined as

$$\mathcal{P}_{t,n} = \left\{ x \in [0, 1]^n : \sum_{i=1}^{n} x_i \leq t \right\}. \tag{2}$$

Notice that for $t = 1$ we obtain a simplex and for $t = n/2$ we obtain half of the hypercube. Moreover different values to t yield very different properties regarding approximability using sparse inequalities, as discussed in [2].

Proposition 2. *The following hold:*

1. $d(\mathcal{P}_{1,n}, \mathcal{P}_{1,n}^k) = \frac{\sqrt{n}}{k} - \frac{1}{\sqrt{n}}$.
2. $d(\mathcal{P}_{n/2,n}, \mathcal{P}_{n/2,n}^k) = \begin{cases} \sqrt{n}/2 & \text{if } k \leq n/2 \\ \frac{n\sqrt{n}}{2k} - \frac{\sqrt{n}}{2} & \text{if } k > n/2 \end{cases}$.
3. $\mathcal{P}_{t,n}^k = [0, 1]^n$ *for all* $t \leq n$ *and* $k \leq t$.

We will also consider symmetrized versions of the polytopes $\mathcal{P}_{t,n}$. To define this symmetrization, for $x \in \mathbb{R}^n$ and $I \subset [n]$ let x^I denote the vector obtained by switching the sign of the components of x not in I:

$$x_i^I = \begin{cases} x_i & \text{if } i \in I \\ -x_i & \text{if } i \notin I. \end{cases}$$

More generally, for a set $P \subseteq \mathbb{R}^n$ we define $P^I = \left\{ x^I \in \mathbb{R}^n : x \in P \right\}$.

Definition 3. For a polytope $P \subseteq \mathbb{R}_+^n$, we define its symmetrized version $\overline{P} = \operatorname{conv}\left(\bigcup_{I \subseteq [n]} P^I \right)$.

Note that $\overline{\mathcal{P}_{1,n}}$ is the cross polytope in dimension n; more generally, we have the following external description of the symmetrized versions of $\mathcal{P}_{t,n}$ and $\mathcal{P}_{t,n}^k$ (proof presented in Appendix B).

Lemma 4.

$$\overline{\mathcal{P}_{t,n}} = \left\{ x \in [-1, 1]^n : \forall I \subset [n], \sum_{i \in I} x_i - \sum_{i \in [n] \setminus I} x_i \leq t \right\} \tag{3}$$

$$\overline{\mathcal{P}_{t,n}}^k = \left\{ x \in [-1, 1]^n : \forall I \in \binom{[n]}{k}, \forall I^+, I^- \text{ partition of } I, \sum_{i \in I^+} x_i - \sum_{i \in I^-} x_i \leq t \right\}. \tag{4}$$

3. Main results

In our first result (Section 4), we point out that in the worst case LP relaxations plus sparse inequalities provide a very weak approximation of the integer hull.

Theorem 5. *For every even integer n there is a polytope $Q_n \subseteq [0, 1]^n$ such that:*

1. $\mathcal{P}_{n/2,n} = \operatorname{conv}(Q_n \cap \mathbb{Z}^n)$
2. $d(\mathcal{P}_{n/2,n}, (\mathcal{P}_{n/2,n})^k \cap Q_n) = \Omega\left(\sqrt{n}\right)$ *for all* $k \leq n/2$.

In Section 5 we consider the second question: How well does the approximation improve if we allowed a budgeted number of dense valid inequalities. Notice that for the polytope $\mathcal{P}_{\frac{n}{2},n}$, while Proposition 2 gives that $d(\mathcal{P}_{\frac{n}{2},n}, \mathcal{P}_{\frac{n}{2},n}^k) \geq \Omega(\sqrt{n})$, adding exactly *one* dense cut ($ex \leq n/2$) to the k-sparse closure (even for $k = 1$) would yield the original polytope $\mathcal{P}_{\frac{n}{2},n}$.

We consider instead the symmetrized polytope $\overline{\mathcal{P}_{\frac{n}{2},n}}$. Notice that while this polytope needs 2^n dense inequality to be described *exactly*, it could be that a small number of dense inequalities, together with sparse cuts, is already enough to provide a good approximation; we observe that in higher dimensions valid cuts for $\overline{\mathcal{P}_{\frac{n}{2},n}}$ can actually cut off significant portions of $[-1, 1]^n$ in *multiple orthants*. Nevertheless, we show that *exponentially* many dense inequalities are required to improve the approximation significantly.