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Interior-point algorithm for linear optimization based on a new trigonometric kernel function

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ABSTRACT

In this paper, we present a new primal–dual interior-point algorithm for linear optimization based on a trigonometric kernel function. By simple analysis, we derive the worst case complexity for a large-update primal–dual interior-point method based on this kernel function. This complexity estimate improves a result from El Ghami et al. (2012) and matches the one obtained in Reza Peza Peyghami et al. (2014).

1. Introduction

After the landmark paper of Karmarkar [4], linear optimization (LO) became an active area of research, due to their wide applications in the real world problems. The resulting interior-point methods (IPMs) are now among the most effective methods for solving LO problems. A number of various IPMs has been proposed and analyzed. For these the reader refers to [11,12,5,6,2]. The primal–dual IPMs for LO problems were firstly introduced by Megiddo [11].

Peng et al. [7] introduced a class of "self-regular kernel functions" and designed primal–dual IPMs based on this class of functions for LO and SDO. They obtained $O(\sqrt{n}\log n\log(n/\varepsilon))$ complexity bound for large-update primal–dual IPMs for LO. Later on, Qian et al. [9] proposed a new kernel function with simple algebraic expression for SDO and established the iteration complexity as $O(n^{3/4}\log(n/\varepsilon))$. Recently, M. El Ghami et al. [3] presented a large-update IPM based on a kernel function with a trigonometric barrier term for LO and obtained the same iteration bound with [9]. Very recently, M. Reza Peyghami et al. [10] proposed a large-update IPM based on a trigonometric kernel function and derived the polynomial complexity enjoys $O(n^{2/3}\log(n/\varepsilon))$, which improved the complexity result for trigonometric kernel function than [9].

Motivated by their work, in this paper we introduce a new trigonometric kernel function (neither self-regular function nor the function that [3,10] proposed) and propose a IPM for LO based on this kernel function. We develop some new analytic tools that are used in the complexity analysis of the algorithm. Finally, we obtain the same complexity result with [1] for the large-update primal–dual IPM.

The paper is organized as follows. In Section 2, we briefly recall the basic concepts of IPMs for LO. The generic primal–dual IPM for LO is presented in Section 3. In Section 4, we introduce a new kernel function and study its properties. Finally, we analyze the algorithm and obtain the worst case complexity result in Section 5.

2. Preliminaries

In this section, we briefly recall the basic concepts of IPMs for LO. The standard LO problem is as follows

$$(P)\quad \min\,\{c^Tx : Ax = b, x \geq 0\},$$

where $A \in \mathbb{R}^{m\times n}$ with $rank(A) = m \leq n, x, c \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$. The dual problem of (P) is given by

$$(D)\quad \max\,\{b^Ty : A^Ty + s = c, s \geq 0\},$$

where $y \in \mathbb{R}^m$ and $s \in \mathbb{R}^n$. Without loss of generality, we may assume that the problems (P) and (D) satisfy the interior-point condition (IPC) [4], i.e., there exist x^0 and (y^0, s^0) such that

$$Ax^0 = b,\quad x^0 > 0,\qquad A^Ty^0 + s^0 = c,\quad s^0 > 0.$$

It is well known that finding an optimal solution of (P) and (D) is equivalent to solving the following system

$$Ax = b,\quad x \geq 0,\qquad A^Ty + s = c,\quad s \geq 0,\qquad xs = 0. \tag{1}$$

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The basic idea of primal–dual IPMs is to replace the third equation in (1) by a parametric equation $xs = \mu e$, where μ is a positive parameter, i.e.,

$$Ax = b, \quad x \geq 0, \qquad A^T y + s = c, \quad s \geq 0, \qquad xs = \mu e. \tag{2}$$

Surprisingly enough, if the IPC is satisfied, the parameterized system (2) has a unique solution, for each $\mu > 0$. It is denoted as $(x(\mu), y(\mu), s(\mu))$ and we call $x(\mu)$ the μ-center of (P) and $(y(\mu), s(\mu))$ the μ-center of (D). The set of μ-centers (with μ running through all positive real numbers) gives a homotopy path, which is called the central path of (P) and (D). The relevance of the central path for LO was recognized first by Sonnevend [12] and Megiddo [5]. If $\mu \to 0$, then the limit of the central path exists and since the limit points satisfy the complementarity condition, the limit yields optimal solutions for (P) and (D).

For fixed $\mu > 0$, a direct application of the Newton method to the system (2), we have the following system

$$A\Delta x = 0, \qquad A^T \Delta y + \Delta s = 0, \qquad s\Delta x + x\Delta s = \mu e - xs. \tag{3}$$

Since A has full row rank, the system (3) has a unique solution $(\triangle x, \triangle y, \triangle s)$ which defines the search direction. By taking a step along the search direction, one constructs a new iterate point

$$x_+ := x + \alpha\triangle x, \qquad y_+ := y + \alpha\triangle y, \qquad s_+ := s + \alpha\triangle s, \tag{4}$$

where $\alpha \in (0, 1]$ is obtained by using some rules so that the new iterate satisfies $(x_+, y_+, s_+) > 0$.

For the motivation of the new method, let us define the scaled vector v as

$$v := \sqrt{xs/\mu}.$$

Note that the pair (x, s) coincides with the μ-center $(x(\mu), s(\mu))$ if and only if $v = e$. Using the scaled vector v, the Newton system (3) can be rewritten as

$$\bar{A}d_x = 0, \qquad \bar{A}^T \triangle y + d_s = 0, \qquad d_x + d_s = v^{-1} - v, \tag{5}$$

where

$$\bar{A} := (1/\mu)AV^{-1}X = AS^{-1}V, \qquad d_x = v\triangle x/x, \qquad d_s = v\triangle s/s. \tag{6}$$

A crucial observation is that the right hand side $v^{-1} - v$ in the third equation of (5) equals minus gradient of the barrier function $\Psi(v) = \sum_{i=1}^{n} \psi_c(v_i)$, $\psi_c(t) = (t^2 - 1)/2 - \log t$, for $t > 0$, it can be easily seen that $\psi_c(t)$ is a strictly differentiable convex function on $\mathbb{R}^n_{++}$ with $\psi_c(e) = \psi_c{}'(e) = 0$, i.e., it attains its minimal value at $t = e$. In this paper, we replace the barrier function $\Psi_c(v)$ by a barrier function $\Psi(v) = \sum_{i=1}^{n} \psi(v_i)$, where $\psi(v)$ is any strictly differentiable convex function on $\mathbb{R}^n_{++}$ with $\psi(e) = \psi'(e) = 0$, the system (5) is converted to the following system

$$\bar{A}d_x = 0, \qquad \bar{A}^T \triangle y + d_s = 0, \qquad d_x + d_s = -\nabla\Psi(v). \tag{7}$$

3. A generic primal–dual interior-point algorithm

The generic form of the algorithm is shown in Fig. 1.

Remark 1. The choice of the barrier update parameter θ plays an important role in both theory and practice of IPMs. Usually, if θ is a constant independent of the dimension n of the problem, for instance, $\theta = 1/2$, then we call the algorithm a large-update (or long-step) method. If θ depends on the dimension of the problem, such as $\theta = 1/\sqrt{n}$, then the algorithm is called a small-update (or short-step) method.

Remark 2. The choice of the step size α ($\alpha > 0$) is another crucial issue in the analysis of the algorithm. In the theoretical analysis the step size α is usually given a value that depends on the closeness of the current iterates to the μ-center. Hence it has to be made sure that the closeness of the iterates to the current μ-center improves by a sufficient amount.

```
Algorithm 1
Input:
   A barrier function Ψ(v);
   a threshold parameter τ > 0;
   a barrier update parameter θ, 0 < θ < 1;
   an accuracy parameter ε > 0;
begin
   x := e;  s := e;  μ := 1;
   while nμ > ε do
   begin
      μ := (1 − θ)μ;
      while Ψ(v) > τ do
      begin
         x := x + α△x;  y := y + α△y;  s := s + α△s;
         v := √(xs/μ);
      end
   end
end
```

Fig. 1. Generic primal–dual algorithm for LO.

4. The new kernel function and its properties

This section is devoted to introduce a new kernel function and study its properties, which are used in the complexity analysis of Algorithm 1.

In this paper, we consider a new univariate function as follows

$$\psi(t) = (t-1)^2/2t + (t-1)^2/2 + \tan^2 h(t)/8, \tag{8}$$

where

$$h(t) = \pi(1-t)/(4t+2). \tag{9}$$

This kernel function has a trigonometric term which differs from the one proposed in [9] and from the one proposed by [3,10]. The first three derivatives of the function $\psi(t)$ are

$$\psi'(t) = (2t^3 - t^2 - 1)/2t^2 + h'(t)\tan h(t)(1 + \tan^2 h(t))/4,$$
$$\psi''(t) = (1+t^3)/t^3 + (1+\tan^2 h(t)) \times [h''(t)\tan h(t) + h'^2(t)(1 + 3\tan^2 h(t))]/4,$$
$$\psi'''(t) = -3/t^4 + (1+\tan^2 h(t))k(t)/4,$$

where

$$h'(t) = -6\pi/(2+4t)^2 < 0, \qquad h''(t) = 48\pi/(2+4t)^3 > 0,$$
$$h'''(t) = -576\pi/(2+4t)^4 < 0,$$
$$k(t) := 3h'(t)h''(t)(1+3\tan^2 h(t)) + 4h'^3(t)\tan h(t)(2+3\tan^2 h(t)) + h'''(t)\tan h(t)).$$

In order to study the properties of our kernel function, we need the following technical lemmas.

Lemma 4.1 (*Lemma 2.1 in [3]*)**.** *For the function $h(t)$ defined in (9), one has*

$$\tan h(t) - 1/(3\pi t) > 0, \quad 0 < t \leq 1/2. \tag{10}$$

Lemma 4.2. *Let $\psi(t)$ be as defined in (8), then*

(i) $\psi''(t) > 1, \ \forall t > 0,$ (11)

(ii) $t\psi''(t) + \psi'(t) > 0, \ \forall t > 0,$ (12)

(iii) $t\psi''(t) - \psi'(t) > 0, \ \forall t > 0,$ (13)

(iv) $\psi'''(t) < 0, \ \forall t > 0.$ (14)

Proof. The detailed proof see http://wenku.baidu.com/view/8a3c985033d4b14e84246833 or see Appendix. □