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An infeasibility certificate for nonlinear programming based on Pareto criticality condition

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ABSTRACT

This paper proposes a new necessary condition for the infeasibility of nonlinear optimization problems, that becomes also sufficient under a convexity assumption, which is stated as a Pareto-criticality condition of an auxiliary multi-objective optimization problem. This condition is evaluated in a search that either leads to a feasible point or to a point at which the infeasibility conditions hold. The resulting infeasibility certificate has global validity in convex problems and has at least a local meaning in generic nonlinear problems.

1. Introduction

Certificates of infeasibility can be useful within optimization algorithms in order to allow the fast determination of the inconsistency of the problem constraints, avoiding spending large computational times in infeasible problems, and also providing a guarantee that a problem is indeed not solvable. A series of results in interior-point based linear programming has been related to the construction of infeasibility certificates [4]. The issue of detecting infeasibility in optimization problems has been particularly important in the context of mixed integer linear programming [2]. In recent convex analysis literature, some infeasibility certificates have been derived for conic programming [12,4] and for the monotone complementarity problem [3]. This last result has been extended to general convex optimization problems [1]. More general studies involving nonlinear programming were presented in [6] and [13]. Recently, an augmented Lagrangian method has been introduced to detect infeasibility [5,9].

The main purpose of this paper is to characterize infeasibility of nonlinear optimization problems as a Pareto-criticality condition of an auxiliary problem. We show a structural similarity between the Kuhn–Tucker conditions for Efficiency (KTE) and a new necessary condition for infeasibility (INF), which also becomes sufficient under the assumption of strict convexity. The application of the proposed certificate is straightforward even in the case of generic nonlinear functions, without the assumption of convexity. In that case, the certificate has local meaning only. The infeasibility condition proposed in this paper is a new infeasibility certificate in finite-dimensional spaces.

The proposed procedure is composed of the following steps: (i) An auxiliary unconstrained multi-objective optimization problem is defined. (ii) A Pareto-critical point of this auxiliary problem is determined. (iii) This point is either a feasible point of the original problem or a point at which the (INF) condition holds. (iv) If (INF) holds on some point, a necessary infeasibility condition is established (this condition is also sufficient in convex problems). Such a verification is straightforward, leading to a potentially useful infeasibility certificate. The key issue that is exploited in the proposed procedure is the fact that the regions where the (INF) condition reaches criticality are often relatively large. This means that points in those regions can be found using numerical algorithms which perform inexact searches, requiring low computational costs. No previous certificate was based on the search for points belonging to this specific region. This means that, even though other methods may also present good computational behavior, the present method expands the repertoire of principles that can be used for the purpose of supporting infeasibility certificates.

The remainder of the paper is organized as follows: Section 2 reviews the background material. Section 3 presents the infeasibility

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condition (INF). Section 4 presents the algorithms used in testing the (INF) condition. Section 5 presents the computational experiments.

2. Background

2.1. Notation

Most of the notation adopted here is rather standard. In particular, the following operators, applied to vector arguments, mean:

($\leq$) Each coordinate of the first argument is less than or equal to the corresponding coordinate of the second argument;
($<$) Each coordinate of the first argument is smaller than the corresponding coordinate of the second argument;
($\prec$) Each coordinate of the first argument is less than or equal to the corresponding coordinate of the second argument, and at least one coordinate of the first argument is strictly smaller than the corresponding coordinate of the second argument.

Similarly the operators ($\geq$), ($>$) and ($\succ$) can be defined in the analogous way. Also $F(\bar{x}) = \left[\nabla f_1(\bar{x})\ \nabla f_2(\bar{x}) \cdots \nabla f_p(\bar{x})\right]$ denotes the Jacobian matrix of the function $f(\cdot) : \mathbb{R}^n \mapsto \mathbb{R}^p$ at the point $\bar{x}$ and $G(\bar{x}) = [\nabla g_1(\bar{x})\ \nabla g_2(\bar{x}) \cdots \nabla g_m(\bar{x})]$ denotes the Jacobian matrix of the function $g(\cdot) : \mathbb{R}^n \mapsto \mathbb{R}^m$ at the point $\bar{x}$.

2.2. Constrained optimization

Consider the optimization problem defined by:

$$\begin{aligned} &\min_x f(x) \\ &\text{subject to} : g(x) \leq 0 \end{aligned} \tag{1}$$

in which $f(\cdot) : \mathbb{R}^n \mapsto \mathbb{R}^p$ and $g(\cdot) : \mathbb{R}^n \mapsto \mathbb{R}^m$ are vector functions. The set of feasible points for this problem is denoted by

$$\Omega \triangleq \left\{x \in \mathbb{R}^n \mid g(x) \leq 0\right\}. \tag{2}$$

In the particular case of $p = 1$ the problem (1) becomes a single-objective optimization problem and when $p > 1$ the problem becomes multi-objective. In the last case, a feasible point $x \in \mathbb{R}^n$ of the decision variable space is said to be dominated by another feasible point $\bar{x} \in \mathbb{R}^n$ if $f(\bar{x}) \prec f(x)$. The solution set of the multi-objective optimization problem is defined as the set $\mathcal{P} \subset \Omega$ of feasible points that are not dominated by any other feasible point. This set is called the *efficient solution set*, or the *Pareto-optimal set*. In order to state general results, the solution set of a single-objective problem is also denoted by $\mathcal{P}$.

The following compactness condition is assumed here:

Assumption 2.1. Assume that there is a subset of the constraint functions, $\{g_1(\cdot), g_2(\cdot), \ldots, g_k(\cdot)\}$, with $k \leq m$, such that the set $\Omega_c \subset \mathbb{R}^n$: defined by

$$\Omega_c = \{x \mid g_1(x) \leq 0,\ g_2(x) \leq 0, \ldots, g_k(x) \leq 0\}$$

is a non-empty compact set. ◇

This assumption holds in a large class of problems, for instance when there is a "box" in the decision variable space where the search is to be conducted.

2.3. Pareto optimality

In multi-objective optimization, the main focus is on producing a trade-off solution set representing the best possible compromises between different (usually conflicting) objectives. Thus, in order to adopt a suitable concept of optimality, the Pareto-optimality is used:

Definition 2.1. Let $\Omega \subset R^n$ be a non-empty set of feasible solutions and $f(\cdot) : \mathbb{R}^n \mapsto \mathbb{R}^p$ be a vector function. A feasible solution $\bar{x} \in \Omega$ is called a Pareto-optimal solution of the multi-objective optimization problem (1) if and only if there does not exist any $x \in \Omega$ such that $f(x) \prec f(\bar{x})$.

2.4. Kuhn–Tucker conditions for efficiency

Let $\lambda \in \mathbb{R}^p$ and $\mu \in \mathbb{R}^m$. The Kuhn–Tucker necessary conditions for the efficiency of a solution $\bar{x}$ to problem (1) are stated as [7,10]:

$$\text{(KTE)} \begin{cases} F(\bar{x})\lambda + G(\bar{x})\mu = 0 \\ \lambda \succ 0, \quad \mu \geq 0 \\ g(\bar{x}) \leq 0 \\ \mu_i g_i(\bar{x}) = 0; \quad \forall\, i = 1, \ldots, m. \end{cases} \tag{3}$$

Notice that the Karush–Kuhn–Tucker conditions for optimality of the single-objective case are a particular case of KTE.

3. Conditions of infeasibility

For problem (1), given a point $\bar{x} \in \mathbb{R}^n$, one of the four possibilities below must happen (by exhaustion):

(a) $\bar{x} \in \mathcal{P}$, which means that the Kuhn–Tucker necessary conditions for Efficiency (KTE) must hold.
(b) $\bar{x} \in \Lambda$, with Λ defined as the set of points for which the infeasibility conditions (INF) hold:

$$\text{(INF)} \begin{cases} \exists i \mid g_i(\bar{x}) > 0 \\ G(\bar{x})\mu = 0 \\ \mu \succ 0 \\ g_j(\bar{x}) < 0 \Rightarrow \mu_j = 0 \end{cases} \tag{4}$$

for some vector of multipliers $\mu \in \mathbb{R}^m$.
(c) $\bar{x} \in \Omega$ and $\bar{x} \notin \mathcal{P}$.
(d) $\bar{x} \notin \Omega$ and $\bar{x} \notin \Lambda$.

Points that satisfy the (KTE) conditions are Pareto-critical for problem (1). The condition (INF) is very similar to (KTE). As will be shown here, the points that satisfy (INF) conditions are also Pareto-critical w.r.t. another auxiliary problem. Define the vector function $\hat{g}(\cdot) : \mathbb{R}^n \mapsto \mathbb{R}^m$ as:

$$\hat{g}_i(x) = \begin{cases} 0, & \forall\, x \mid g_i(x) \leq 0 \\ g_i(x), & \forall\, x \mid g_i(x) > 0 \end{cases} \quad i = 1, \ldots, m. \tag{5}$$

From the above definition, the following auxiliary problem is defined:

$$\min_x \hat{g}(x). \tag{6}$$

The corresponding efficient solution set of this problem is denoted by $\mathcal{A}$:

$$\mathcal{A} = \left\{x \in \mathbb{R}^n \mid \nexists\, \bar{x} \in \mathbb{R}^n \text{ such that } \hat{g}(\bar{x}) \prec \hat{g}(x)\right\}. \tag{7}$$

It should be noticed that under Assumption 2.1, it can be stated that: $\mathcal{A} \neq \emptyset$ and $\mathcal{A} \subset \Omega_c$. Denote by $\hat{g}(\mathcal{A})$ the image set of function $\hat{g}(\cdot)$ over $\mathcal{A}$. The following lemma comes directly from the definition of the function $\hat{g}(\cdot)$:

Lemma 3.1. *The following statements hold:*

(i) $\Omega \neq \emptyset \Rightarrow \hat{g}(\mathcal{A}) \equiv 0,\ \Omega \equiv \mathcal{A}$
(ii) $\Omega = \emptyset \Rightarrow \hat{g}(x) \succ 0\ \forall\, x \in \mathcal{A}$. ◇

Proof. Statement (i) comes from the fact that if the problem is feasible ($\Omega \neq \emptyset$), then each function $\hat{g}_i(x)$ will reach its minimum on $\hat{g}_i(x) = 0$ for every $x \in \Omega$. Therefore, the Pareto-set $\mathcal{A}$ of the auxiliary problem (6) is identical to Ω, i.e., $\Omega \equiv \mathcal{A}$, and $\hat{g}(\mathcal{A}) \equiv 0$.