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A value for games on augmenting systems with a coalition structure

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ABSTRACT

The purpose of this paper is to discuss games on augmenting systems with a coalition structure, where all feasible subsets of the coalition structure and that of each union both form an augmenting system. A value named the augmenting symmetric Banzhaf coalitional value is introduced. The distribution of power in the Basque Parliament emerging is provided to illustrate the concrete application of the value. Meanwhile, two axiomatic systems are established.

1. Introduction

Games with a coalition structure, as an important form of cooperation, are first introduced by Aumann and Dréze [5]. In this model [5], the coalitions are independent with each other. Later, Owen [13,14] further discussed games with a coalition structure, where the probability of cooperation among coalitions is considered. Two payoff indices are defined, which are named the Owen value and the Banzhaf–Owen value. Recently, Alonso-Meijide and Fiestras-Janeiro [4] defined another payoff index called the symmetric Banzhaf value. It calculates the total payoff that a union can obtain by means of the Banzhaf value and then, for each union its total allocation is split among its members using the Shapley value.

Different to games with a coalition structure, there is another kind of games, where players' payoffs are relevant to their orders in the cooperation. Myerson [12] first considered this problem and introduced games with communication situations. Bilbao [6] studied games on convex geometries and defined the Shapley value for this kind of games. Algaba et al. [1,2] discussed games on antimatroids and researched the Shapley value for this class of games. Bilbao [8] introduced games on augmenting systems and discussed the relationship between augmenting system, antimatroid and convex geometry. From the discussion, we know that antimatroid and convex geometry are two special cases of augmenting system.

Vázquez-Brage et al. [15] studied games with a priori union and graph restricted communication. The Owen graph value for this family of games is proposed. An axiomatic system is proposed by modifying the properties considered in the Owen and Myerson values. Alonso-Meijide et al. [3] further researched this family of games and defined two new coalition values named the Banzhaf–Owen graph value and the symmetric Banzhaf graph value. Meng and Zhang [10] introduced games on convex geometries with a coalition structure and defined the generalized Owen value.

Based on researches about games with a coalition structure and games on augmenting systems, we here study games on augmenting systems with a coalition structure, where all feasible subsets of the coalition structure and that of each union both form an augmenting system. For this family of games, the augmenting symmetric Banzhaf coalitional value is defined, and two axiomatic characterizations are examined. From the relationship between augmenting system and convex geometry, we know that games on convex geometries with a coalition structure [10] are a special case of games on augmenting systems with a coalition structure.

2. Preliminaries

2.1. Cooperative games with a coalition structure

Let $N = \{1, 2, \ldots, n\}$ be the player set, and $P(N)$ be the power set of N. A coalition structure $\Gamma = \{B_1, B_2, \ldots, B_m\}$ on player set

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N is a partition of N, i.e., $\bigcup_{1\le h\le m} B_h = N$ and $B_h \bigcap B_l = \varnothing$ for all $h, l \in M = \{1, 2, \ldots, m\}$ with $h \neq l$, which is denoted by (N, Γ). Let

$$L(N,\Gamma) = \left\{ S | S = \bigcup_{q\in R} B_q \cup T, \forall R \subseteq M \setminus k (k \in M), \forall T \subseteq B_k \right\}.$$

The cardinality of any $S \in L(N, \Gamma)$ is denoted by the corresponding low case s. A game with a coalition structure is a set function $v : L(N, \Gamma) \to \Re_+$ such that $v(\varnothing) = 0$. The set of all games with a coalition structure is denoted by $G(N, \Gamma)$. To denote simply, we will omit braces for singletons, e.g. by writing $\varnothing, i, k$ instead of $\{\varnothing\}$, $\{i\}$ and $\{k\}$ for any $\{i\} \subseteq N$ and any $\{k\} \subseteq M$.

Definition 1 (*[13]*). Let $v \in G(N, \Gamma)$, if we have $v^B(R) = v(\cup_{r\in R} B_r)$ for any $R \subseteq M$, then v^B is said to be a quotient game on (N, Γ), where Γ and M as shown above, denoted by (M, v^B).

Alonso-Meijide and Fiestras-Janeiro [4] defined the symmetric Banzhaf value on $G(N, \Gamma)$ as follows:

$$\beta_i(N, v, \Gamma) = \sum_{R\subseteq M\setminus k} \sum_{T\subseteq B_k \setminus i} \frac{1}{2^{m-1}} \frac{r!(b_k - t - 1)!}{b_k!} \times \left(v\left(Q \bigcup T \bigcup i\right) - v\left(Q \bigcup T\right)\right) \quad \forall i \in N, \tag{1}$$

where $Q = \bigcup_{q\in R} B_q$, and $B_k \in \Gamma$ is an union such that $i \in B_k$.

2.2. Cooperative games on augmenting systems

A set system on N is a pair $(N, \mathscr{F})$, where $\mathscr{F} \subseteq 2^N$ is a family of subsets. The sets belong to $\mathscr{F}$ are called feasible sets.

Definition 2 (*[7]*). An augmenting system is a set system $(N, \mathscr{F})$ with the following properties:

A1: $\varnothing \in \mathscr{F}$;
A2: If $S, T \in \mathscr{F}$ with $S \bigcap T \neq \varnothing$, then $S \bigcup T \in \mathscr{F}$;
A3: If $S, T \in \mathscr{F}$ with $S \subseteq T$, then there exists $i \in T \setminus S$ such that $S \bigcup i \in \mathscr{F}$.

An augmenting system $(N, \mathscr{F})$ is said to be normal, if $N = \bigcup_{S\in\mathscr{F}} S$. For a normal augmenting system, Bilbao and Ordoñez [8] introduced the following concepts:

A compatible ordering of an augmenting system $(N, \mathscr{F})$ is given by $i_1 < i_2 < \cdots < i_n$ such that $\{i_1, i_2, \ldots, i_j\} \in \mathscr{F}$ for all $j = 1, 2, \ldots, n$. Thus, a compatible ordering of $(N, \mathscr{F})$ corresponds to a maximal chain in $\mathscr{F}$. The set of all maximal chain in $\mathscr{F}$ is denoted by $\mathrm{Ch}(\mathscr{F})$, and the cardinality of $\mathrm{Ch}(\mathscr{F})$ is denoted by $c(N)$. For any $S \in \mathscr{F}$, $c(S)$ is the number of maximal chains from $\varnothing$ to S, and $c(S \bigcup i, N)$ is the number of maximal chains from $S \bigcup i$ to N, where $S \bigcup i \in \mathscr{F}$. Given an element $i \in N$ and a compatible ordering $C \in \mathrm{Ch}(\mathscr{F})$, let $C(i) = \{i \text{ is the last element in } C\}$. A game on an augmenting system is a set function $v : \mathscr{F} \to \Re_+$, such that $v(\varnothing) = 0$. The set of all games on the augmenting system $\mathscr{F}$ is denoted by $G(N, \mathscr{F})$.

Let $S \in \mathscr{F}$, $i \in N \setminus S$ is called an augmenting point of S if $S \bigcup i \in \mathscr{F}$. The set of all augmenting points of S is denoted by $S^* = \{i \in N \setminus S : S \bigcup i \in \mathscr{F}\}$.

Bilbao [8] introduced the Shapley value on $G(N, \mathscr{F})$ as follows:

$$\phi_i(N, v, \mathscr{F}) = \sum_{\{S\in\mathscr{F}: i\in S^*\}} \frac{c(S)c\left(S \bigcup i, N\right)}{c(N)} \times \left(v\left(S \bigcup i\right) - v(S)\right) \quad \forall i \in N. \tag{2}$$

3. Cooperative games on augmenting systems with a coalition structure

3.1. Basic concepts

Let $\Gamma = \{B_1, B_2, \ldots, B_m\}$ be a coalition structure on player set N. From Definition 2, we know when the domain of N is restricted in the union B_k, we obtain an augmenting system $(B_k, \mathscr{F}_{B_k})$, where $\mathscr{F}_{B_k} \subseteq 2^{B_k}$ is a family of subsets that satisfies the conditions given in Definition 2.

Similar to the concept of the augmenting system on N, the augmenting system on $M = \{1, 2, \ldots, m\}$ with respect to the coalition structure $\Gamma = \{B_1, B_2, \ldots, B_m\}$ is a set system $(M, \mathscr{F}_M)$ such that P1 : $\varnothing \in \mathscr{F}_M$; P2: If $H, K \in \mathscr{F}_M$ with $H \bigcap K \neq \varnothing$, then $H \bigcup K \in \mathscr{F}_M$; P3: If $H, K \in \mathscr{F}_M$ with $H \subseteq K$, then there exists $p \in K \setminus H$ such that $H \bigcup p \in \mathscr{F}_M$.

Games on augmenting systems with a coalition structure mean that all feasible subsets of M and that of each $B_k \in \Gamma$ respectively form an augmenting system, denoted by $(N, \Gamma, \mathscr{F})$.

Let $L(N, \Gamma, \mathscr{F}) = \{S | S = \bigcup_{q\in R} B_q \cup T, \forall R \in \mathscr{F}_M \setminus \{k\} : R \cup k \in \mathscr{F}_M (k \in M), \forall T \in \mathscr{F}_{B_k}\}$. By $G(N, \Gamma, \mathscr{F})$, we denote the set of all games on $(N, \Gamma, \mathscr{F})$. Without special explanation, for any $(N, \Gamma, \mathscr{F})$, we always mean $B_k \in \mathscr{F}_{B_k}$ for any $k \in M$ and $M \in \mathscr{F}_M$.

Following the works of Bilbao and Ordoñez [8] and Faigle and Kern [9], for any $(N, \Gamma, \mathscr{F})$, we define the *hierarchical strength* $h_S^{B_k}(i)$ of $i \in S$ in the feasible coalition $S \in \mathscr{F}_{B_k}$ as follows:

$$h_S^{B_k}(i) = \frac{|\{C \in \mathrm{Ch}(\mathscr{F}_{B_k}) : S \subseteq C(i)\}|}{c(B_k)},$$

where $\mathrm{Ch}(\mathscr{F}_{B_k})$ is the set of all maximal chains in $\mathscr{F}_{B_k}$, and $c(B_k) = |\mathrm{Ch}(\mathscr{F}_{B_k})|$ is the total number of maximal chains in $\mathscr{F}_{B_k}$.

The element $k \in M \setminus H$ is called an augmenting point of H if $H \bigcup k \in \mathscr{F}_M$, the set of all augmenting points of H is denoted by $H^* = \{k \in \mathscr{F}_M \setminus H : H \bigcup k \in \mathscr{F}_M\}$. Let $\xi_k = |\{H \bigcup k \in \mathscr{F}_M : k \in H^*\}|$ and $\xi_k(H) = |\{R \cup k \in \mathscr{F}_M : k \in R^*, H \subseteq R \cup k\}|$.

3.2. The augmenting symmetric Banzhaf coalitional value

Following the works of Alonso-Meijide and Fiestras-Janeiro [4] and Bilbao and Ordoñez [8], we define the augmenting symmetric Banzhaf coalitional value for games on augmenting systems with a coalition structure $G(N, \Gamma, \mathscr{F})$ as follows:

$$\varphi_i(N, v, \Gamma, \mathscr{F}) = \sum_{\{H\in\mathscr{F}_M : k\in H^*\}} \sum_{\{S\in\mathscr{F}_{B_k} : i\in S^*\}} \frac{1}{\xi_k} \times \frac{c(S)c\left(S \bigcup i, B_k\right)}{c(B_k)} \times \left(v\left(Q \bigcup S \bigcup i\right) - v\left(Q \bigcup S\right)\right) \quad \forall \in N, \tag{3}$$

where $Q = \bigcup_{l\in H} B_l$.

Example 1 (*[3]*). The Parliament of the Basque Country, one of Spain's seventeen regions, is constituted by 75 members. Since most decisions are taken by majority, the characteristic function of the game played by the parties with parliamentary representation is as follows, unity for any coalition summing 38 or more members, and zero for the rest. Since elections in 2005, the Parliament has been composed of 22 members of the Basque nationalist conservative party EAJ/PNV, "A", 18 members of the Spanish socialist party PSE-EE/PSOE, "B", 15 members of the Spanish conservative party PP, "C", 9 members of the Basque nationalist left-wing party EHAK/PCTV, "D", 7 members of the Basque nationalist social democrat party EA, "E", 3 members of the Spanish left-wing party EB/IU, "F", and 1 member of the Basque nationalist moderated left-wing party Aralar, "G".