Operations Research Letters 44 (2016) 342–347

Contents lists available at ScienceDirect

Operations Research Letters

journal homepage: www.elsevier.com/locate/orl

Low-order penalty equations for semidefinite linear complementarity problems

Chen Zhao [a,*], Ziyan Luo [b], Naihua Xiu [a]

[a] Department of Mathematics, Beijing Jiaotong University, Beijing 100044, PR China

[b] State Key Laboratory of Rail Traffic Control and Safety, Beijing Jiaotong University, Beijing 100044, PR China

ARTICLE INFO

Article history:
Received 14 September 2015
Received in revised form
7 March 2016
Accepted 7 March 2016
Available online 15 March 2016

Keywords:
Semidefinite linear complementarity problem
Low-order penalty equation
Cartesian P-property
Convergence rate

ABSTRACT

We extend the power penalty method for linear complementarity problem (LCP) (Wang and Yang, 2008) to the semidefinite linear complementarity problem (SDLCP). We establish a family of low-order penalty equations for SDLCPs. Under the assumption that the involved linear transformation possesses the Cartesian P-property, we show that when the penalty parameter tends to infinity, the solution to any equation of this family converges to the solution of the SDLCP exponentially. Numerical experiments verify this convergence result.

1. Introduction

The semidefinite linear complementarity problem (SDLCP), which has provided a unified model for various problems arising from combinatorial optimization [1], engineering [4], system and control theory [3] etc., was initially formulated by Kojima et al. in 1997 [16]. Due to its wide applications in a variety of fields, growing interests are then emerging both in theoretical analysis and computation methods for SDLCPs. From the theoretical perspective, properties on merit functions and complementarity functions [26,28,23], optimality conditions [21], and solution existence and uniqueness [9,10,6,20] of SDLCPs have been extensively investigated. Based upon the achieved theoretical results, concrete numerical methods such as interior-point methods [16,18,19], non-interior methods [7,29], smoothing Newton methods [15,13] etc. for solving SDLCPs have been proposed.

As one of the most important methods for constrained optimization problems, the low-order penalty method has recently received a great deal of attention in solving complementarity problems. For instance, Wang and Yang [27] proposed a power penalty method with the low-order $\ell_{1/k}$ ($k \geq 1$) penalty term for solving the classic linear complementarity problems (LCPs) and established the exponential convergence rate under the cases that the involved coefficient matrix is a nonsingular M-matrix (positive definite with all off-diagonal entries non-positive) or a diagonal matrix. More recently, this power method scheme was further extended to handle more general complementarity problems such as the nonlinear complementarity problems (NCPs) with strong monotonicity [14] and with the uniform P-property [25], the parabolic linear complementarity problem arising from American options pricing problems [22], and the second-order cone complementarity problems (SOCPs) with positive definite linear mappings [11] and with the strong monotone nonlinear mappings [12]. It is well-known that the SOCP is an important extension of classic complementarity problems by substituting the involved nonnegative orthant with the so-called second order cone. As another important extension of classic complementarity problems, the semidefinite complementarity problem (SDCP) is built in the context of positive semidefinite cone. As we have mentioned that the low-order penalty methods have been used very successfully for solving LCPs, NCPs and SOCPs. A natural question then arises: Can we handle the SDCP with the well-performed low-order penalty method? We will answer this question in an affirmative way.

Note that the low-order penalty term in the aforementioned papers mostly takes a specific power penalty form, and most of the conditions to guarantee the solution convergence are somehow very restricted. The other motivation in this paper is to provide a class of low-order penalty approaches under some mild conditions.

* Corresponding author.
E-mail addresses: 14118409@bjtu.edu.cn (C. Zhao), starkeynature@hotmail.com (Z. Luo), nhxiu@bjtu.edu.cn (N. Xiu).

http://dx.doi.org/10.1016/j.orl.2016.03.004

Inspired by the Cartesian product structure which has been widely studied in complementarity problems with different underlying linear spaces [8,11,6], we will consider the following SDLCP with Cartesian products: find a matrix $X \in \mathbb{S}$ such that

$$\mathcal{L}(X) + Q \in \mathbb{S}_+, \quad X \in \mathbb{S}_+, \ \langle \mathcal{L}(X) + Q, X \rangle = 0, \tag{1}$$

where $\mathbb{S}$ denotes the linear space of all $n \times n$ block-diagonal real symmetric matrices with m blocks of size $n_1, \ldots, n_m$, $n = \sum_{i=1}^m n_i$, $\mathbb{S}_+$ denotes the cone of symmetric positive semidefinite matrices in $\mathbb{S}$, $\mathcal{L} : \mathbb{S} \to \mathbb{S}$ is a linear operator and $Q \in \mathbb{S}$ is a given symmetric matrix, $\langle X, Y \rangle$ is the inner product of X and Y, i.e. $\langle X, Y \rangle = \text{trace}(X^T Y)$.

By employing the aforementioned low-order penalty scheme, together with the tool of Löwner operators defined in the symmetric matrix space (see Section 2.2 for details), the following class of nonlinear equations will be employed to approximate the SDLCP (1):

$$\mathcal{L}(X) + Q - \lambda [G(X)]^{\frac{1}{k}} = 0, \tag{2}$$

where $\lambda > 1$ is some parameter, $G : \mathbb{S} \to \mathbb{S}$ is a Löwner operator (see Definition 2.2) generated by some real-valued function $g : \mathbb{R} \to \mathbb{R}$ satisfying the following conditions:

(a) $g(t)$ is monotonically non-increasing and continuous;
(b) $g(t) = 0$ if $t \geq 0$;
(c) $g(t) \geqslant -t$ if $t \in [-a, 0)$, with some positive scalar a.

By taking the advantage of the Cartesian P-property introduced by Chen and Qi in [6], we will establish the convergence of the proposed class of low-order penalty methods for Cartesian P-SDLCP with the same exponential convergence rate as shown in the context of LCPs [27], NCPs [14,25] and SOCPs [11,12] when λ tends to infinity. This indicates that the low-order penalty scheme can be successfully extended to handle SDLCPs as well. Besides, the penalty term in our low-order equation (2) apparently contains the aforementioned power penalty form as a special case. In this regard, our model provides more choices for the low-order penalty forms to deal with the complementarity problems. Furthermore, it is worth pointing out that the Cartesian P-property in our model is milder than the strong monotonicity as required in most of the above mentioned references, which shows that our approach can treat a broader class of complementarity problems. All these are our main contributions.

This paper is organized as follows. In Section 2, some preliminary results including the Cartesian P-property and some properties for the SDLCP and Eq. (2) are reviewed, and the definition and the monotonicity of Löwner operators are recalled and developed for sequential analysis. In Section 3, properties of solutions to the proposed nonlinear matrix equation are proposed and the convergence analysis of the proposed class of low-order penalty methods is established. In Section 4, we carry out some numerical experiments in order to verify our convergence results.

Notations which will be used throughout the paper are introduced here. For a matrix $A \in \mathbb{S}$, $A_\nu \in \mathbb{R}^{n_\nu \times n_\nu}$ denotes the νth block of A ($\nu = 1, \ldots, m$). $\| \cdot \|_F$ denotes the Frobenius-norm on $\mathbb{S}$, i.e. $\|X\|_F = \sum_{\nu=1}^m \|X_\nu\|_F = (\sum_{i=1}^n |\lambda_i|^2)^{1/2}$. $\| \cdot \|_p$ denotes the Schatten p-norm on $\mathbb{S}$, i.e. $\|X\|_p = \sum_{\nu=1}^m \|X_\nu\|_p = (\sum_{i=1}^n |\lambda_i|^p)^{1/p}$. We use $\mathbb{S}_{++}$ to denote the cone of all positive definite matrices in $\mathbb{S}$. When $X \in \mathbb{S}_+$ ($X \in \mathbb{S}_{++}$, respectively), we simply write $X \succcurlyeq 0$ ($X \succ 0$, respectively).

2. Preliminary results

To begin, we first recall some related preliminary definitions and properties.

2.1. The SDLCP

Analogous to the case of classic linear complementarity problem, the SDLCP can be equivalently converted into some variational inequality problem over the positive semidefinite matrix cone $\mathbb{S}_+$. Considering the Cartesian product structure of the underlying $n \times n$ matrix space with m blocks of size $n_1, \ldots, n_m$, the equivalent reformulation is presented as follows.

Lemma 2.1 (*[8]*). *The SDLCP (1) is equivalent to the following variational inequality problem: find $X \in \mathbb{S}_+$ such that for any $Y \in \mathbb{S}_+$,*

$$\langle Y - X, \mathcal{L}(X) + Q \rangle \geqslant 0,$$

i.e.,

$$\langle Y_\nu - X_\nu, \mathcal{L}_\nu(X) + Q_\nu \rangle \geqslant 0, \quad \forall \nu = 1, \ldots, m, \tag{3}$$

where $\mathcal{L}_\nu(X)$ denotes the νth block of $\mathcal{L}(X)$.

Lemma 2.1 allows us to characterize the solution existence of the original SDLCP by means of that of the variational inequality problem (3). In this vein, the following lemma for the mentioned solution existence is presented, which is extended from [8, Proposition 3.5.1] by substituting $\mathbb{R}^n$ with $\mathbb{S}$.

Lemma 2.2. *Let $\mathcal{K} \subseteq \mathbb{S}$ be a closed, convex set and $\mathcal{F} : \mathbb{S} \to \mathbb{S}$ be continuous. If there exists a matrix $X^{ref} \in \mathcal{K}$ such that the set*

$$L'_{\leqslant} := \{X \in \mathcal{K} : \max_{1 \leqslant \nu \leqslant m} \langle X_\nu - X_\nu^{ref}, F_\nu(X) \rangle \leqslant 0\},$$

is bounded, then the solution set of the variational inequality problem: find $X \in \mathcal{K}$, such that, for all $Y \in \mathcal{K}$,

$$\langle Y - X, \mathcal{L}(X) + Q \rangle \geqslant 0,$$

is nonempty and compact.

Besides the solution existence of the SDLCP, the uniqueness is also very important and has attracted great attention. As a generalization of P-matrices, Chen and Qi [6] introduced the so-called Cartesian P-property for linear transformations which can ensure both the solution existence and solution uniqueness for the corresponding SDLCP.

Definition 2.1 (*[6]*). A linear transformation $\mathcal{L} : \mathbb{S} \to \mathbb{S}$ is said to have the Cartesian P-property if for any $0 \neq X \in \mathbb{S}$

$$\max_{1 \leqslant \nu \leqslant m} \langle X_\nu, \mathcal{L}_\nu(X) \rangle > 0.$$

It is easy to see that when $\mathbb{S}$ contains only one block (i.e., $m = 1$), the Cartesian P-property becomes the strong monotonicity of $\mathcal{L}$, i.e., $\langle X, \mathcal{L}(X) \rangle > 0$ for all $0 \neq X \in \mathbb{S}$, and when $\mathbb{S}$ contains only diagonal matrices (i.e., $m = n$), it becomes the P-property of matrices. Similar to the P-matrix, a linear Cartesian P transformation on $\mathbb{S}$ has the following essential property.

Lemma 2.3 (*[6]*). *A linear transformation $\mathcal{L} : \mathbb{S} \mapsto \mathbb{S}$ has the Cartesian P-property if and only if there exists a constant*

$$\alpha(\mathcal{L}) = \min_{\|X\|_F = 1} \max_{1 \leqslant \nu \leqslant m} \langle X_\nu, \mathcal{L}_\nu(X) \rangle > 0,$$

i.e., there exists a constant $\alpha(\mathcal{L}) > 0$ such that for any $X \in \mathbb{S}$,

$$\max_{1 \leqslant \nu \leqslant m} \langle X_\nu, \mathcal{L}_\nu(X) \rangle \geqslant \alpha(\mathcal{L}) \|X\|_F^2. \tag{4}$$

The solution existence and uniqueness, sometimes called the globally unique solvability (GUS for short) as mentioned in [9], was established by Chen and Qi [6] under the Cartesian P-property.

Theorem 2.1 (*[6]*). *If the linear transformation $\mathcal{L} : \mathbb{S} \to \mathbb{S}$ satisfies the Cartesian P-property, then the SDLCP (1) has a unique solution for any $Q \in \mathbb{S}$.*