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Flexible M/G/1 queueing system with state dependent service rate

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ABSTRACT

In this paper, we propose the extended Poisson distribution which unifies some well-known models in finite queues that have been used to model congestion in vehicular, pedestrian traffic networks as well as the Conway–Maxwell Poisson distribution (Kadane et al., 2006). The main issue is to formulate a M/G/1 queue wherein the service distribution is the Exponential Conway–Maxwell Poisson distribution (Cordeiro et al., 2012) which uses a new defense mechanism against long waiting time. A Bayesian simulation study and an example with a real dataset are provided.

1. Introduction

A major issue while developing queueing systems is to capture in a realistic way the performance of a service center to explain why some queues are long while others are idle. The basic question is: How to reformulate a M/M/1 queueing system to get a steady state, or, some defense mechanism against long waiting time?

In this paper, we develop a new M/M/1 queueing system with dependent service rate to answer this question in which the number of customers in the queue is unknown but follows a unified probability distribution called Extended Conway–Maxwell Poisson distribution. The proposed state distribution includes as particular case the Conway–Maxwell Poisson [5] distribution which includes several well-known models as special case. Moreover, the state probability distribution is a weighted Poisson distribution. So, it is possible to study the behavior of the queue in terms of underdispersion and overdispersion which we believe to be new in this area of research. Other interesting aspect of our approach is the possibility of developing new and flexible models for the service time with state dependent rate in which the server works with a defense mechanism against long waiting times which corresponds to minimal service time among an unknown number of parallel servers and an additional parameter that indicates how the queueing system is affected by the state of the system. In fact, the proposed queueing system is a M/G/1 queue in which the customers follow a Poisson process, while the service time follows an Exponential Conway–Maxwell Poisson distribution as developed by [6]. This M/G/1 queue has an additional parameter, hereafter referred to as the pressure parameter, which indicates how the service time, waiting time and the traffic coefficient are affected by the state of the system. This queue will be obtained by eliminating the unknown number of customers in the likelihood function thereby introducing a new marginal service distribution for the M/G/1 queues, which will be referred to here as the marginal queueing system.

The rest of this paper is organized as follows. In Section 2, we extend the Conway–Maxwell Poisson distribution and then present the mean and variance. In Section 3, some connections to overdispersion and underdispersion are highlighted. In Section 4, we describe the M/M/1 queueing system with state dependent service rate. In Section 5, the equilibrium and waiting time distributions of the proposed queueing system are introduced. In Section 6, the Bayesian inference methodology is formulated for M/ExpCMP/1 queueing system. Simulation studies from the Bayesian point of view and an application to real data set are then presented in Sections 7 and 8, respectively. Finally, Section 9 presents some conclusions and suggestions for future research.

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2. Extended Poisson distribution

The extended Poisson distribution (hereafter denoted by $\text{ExP}(\theta, \phi)$) is a weighted Poisson distribution with reference Poisson parameter θ and weight function $w^\star(m; \phi) = \frac{m!}{\prod_{i=1}^m w(i;\phi)}$, where $w(i; \phi)$ is a positive function with $w(0, \phi) = 1$. Its probability mass function is given by

$$P[M = m; \theta, \phi] = \frac{\theta^m}{\prod_{i=0}^{m} w(i; \phi) K(\theta, \phi)}, \quad m = 0, 1, 2, \ldots, \tag{1}$$

where $K(\theta, \phi) = \sum_{m=0}^\infty \frac{\theta^m}{\prod_{i=1}^m w(i;\phi)}$. The parameter ϕ is an additional quantity which explains how the Poisson distribution is changed to obtain more realistic models. The corresponding probability generating function (pgf) can be easily obtained as

$$A_M(s) = \frac{K(\theta s, \phi)}{K(\theta, \phi)}.$$

Theorem 1. *The $\text{ExP}(\theta, \phi)$ is overdispersed (underdispersed) if the Poisson weight function $m \Longrightarrow w^\star(m, \phi)$ in (1) does not depend on θ and it is logconvex (logconcave).*

Proof. The result follows readily by applying the corollary in [12].

2.1. M/G/c/c state dependent queues

The well known M/G/c/c state-dependent finite queues have been used to model congestion in vehicular and pedestrian traffic networks. The discrete variable M denotes the input Poisson process, G is a general service time distribution with state dependent mean service $E(S)$, and c is the number of parallel servers which also represents the total capacity of the system. The service rate is dependent on the state of the system and some known function $f(m, \phi)$, where $M = m$ is the number of customers in the queueing system. Taking $\theta = \lambda E(S)$ and the weight function

$$w(i, \phi) = \begin{cases} if\ (i, \phi) & : \quad i = 1, 2, \ldots, c \\ 1 & : \quad i = 0 \\ 0 & : \quad i = c, c+1, \ldots, \end{cases} \tag{2}$$

we have $\text{ExP}(\theta, \phi)$ to be the probability of finding m customers in the M/G/c/c queueing system (see, [4]) given by

$$P[M = m] = \frac{(\lambda E(S))^m}{m! \prod_{i=0}^{m} f(i, \phi) K(\theta, \phi)}, \quad m = 0, 1, 2, \ldots, c, \tag{3}$$

where $K(\theta, \phi) = \sum_{m=0}^{c} \frac{(\lambda E(S))^m}{m! \prod_{i=0}^m f(i,\phi)}$.

2.2. Multiserver Queues: M/M/c

We now consider the multiserver M/M/c queueing model in which the number of arriving customers follows a Poisson distribution with rate λ, there are c servers, and each one has an independent and identical exponential service-time distribution with mean $1/\mu$. Let us define the weight function as

$$w(i) = \begin{cases} 1 & : \quad i = 0 \\ i\mu & : \quad i = 1, 2, \ldots, c-1 \\ c\mu & : \quad i = c, c+1, \ldots. \end{cases} \tag{4}$$

Then, the steady-state probability is given by

$$P[M = m] = \begin{cases} \dfrac{\lambda^m [K(\lambda, \mu)]^{-1}}{m! \mu^m} & : \quad m = 0, 1, 2, \ldots, c-1 \\ \dfrac{\lambda^m [K(\lambda, \mu)]^{-1}}{c! \mu^m c^{m-c}} & : \quad m = c, c+1, \ldots, \end{cases} \tag{5}$$

where

$$K(\lambda, \mu) = \sum_{m=0}^{c-1} \frac{\lambda^m}{m! \mu^m} + \sum_{m=c}^{\infty} \frac{\lambda^m}{c^{m-c} c! \mu^m}.$$

2.3. M/M/1 queueing system with general state dependent service rate

The following result extends the Conway–Maxwell Poisson (CMP) distribution proposed by [5].

Theorem 2. *Given a M/M/1 queueing system with state dependent service rate $\mu_m = w(m, \phi)\mu$, where $w(m, \phi)$ is as defined in (1), the probability of having m units in the system, hereafter referred to as the Extended Conway–Maxwell Poisson distribution (in short, $\text{ExCMP}(\rho, \phi)$), is given by*

$$p_w(m; \rho, \phi) = \frac{w^\star(m; \phi) p(m; \rho)}{E_\rho[w^\star(M; \phi)]}$$
$$= \frac{\rho^m}{\prod_{i=0}^{m} w(i; \phi) K_w(\rho, \phi)} \Leftrightarrow w(1; \phi) = 1, \tag{6}$$

where the Poisson weight function is

$$w^\star(m; \phi) = \frac{m!}{\prod_{i=0}^{m} w(i; \phi)}, \tag{7}$$

$$p(m, \rho) = \frac{\rho^m e^{-\rho}}{m!}, \quad m = 0, 1, \ldots, \tag{8}$$

$$K_w(\rho, \phi) = \sum_{m=0}^{\infty} \frac{\rho^m}{\prod_{i=0}^{m} w(i; \phi)}. \tag{9}$$

Proof. The proof is similar to that of [5] by taking $\mu_m = w(m, \phi)\mu$.

Remark 1. Note that the $\text{ExCMP}(\rho, \phi)$ is an Extended Poisson distribution if we take $\theta = \rho$.

Remark 2. Some special cases of the Extended Conway–Maxwell Poisson distributions are as follows:

- Conway–Maxwell Poisson distribution with parameters ρ and ϕ [5]: $w(m, \phi) = m^\phi, m \geq 1$ and $w(0, \phi) = 1$ (denoted by $M \sim \text{CMP}(\rho, \phi)$);
- Poisson distribution with parameter ρ : $w(m, \phi) = m, \ m \geq 1$ and $w(0, \phi) = 1$;
- Geometric distribution with parameter $1 - \rho$: For $w(m, \phi) = 1, \ m \geq 0$, we return to the classical $M/M/1$ queueing system;
- Bardwell–Crow distribution, or, the hyper-Poisson distribution [3], $\text{hP}(\mu, \rho)$, given by

$$P[M = m; \mu, \rho] = \frac{1}{{}_1F_1(1, \mu, \rho)} \frac{\rho^m}{(\mu)_m}, \quad m = 0, 1, 2, \ldots, \tag{10}$$

where $(\mu)_m = \mu(\mu+1)(\mu+2)\ldots(\mu+m-1), \ m \geq 1, \ (\mu)_0 = 1, \ \rho = \frac{\lambda}{\mu}$ and

$${}_1F_1(1, \mu, \rho) = \sum_{m=0}^{\infty} \frac{\rho^m}{(\mu)_m} \tag{11}$$

is the confluent hypergeometric series [10]. The $\text{hP}(\mu, \rho)$ is the $\text{ExCMP}(\rho, \mu)$ if we take

$$w(i, \mu) = \begin{cases} \mu + i - 1 & : \quad i \geq 2 \\ 1 & : \quad i = 1 \\ 1 & : \quad i = 0. \end{cases} \tag{12}$$