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On the sum–product ratio problem and its applications

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ABSTRACT

We study the so-called sum–product ratio problem, under which the numerator is the sum of numbers in a subset and the denominator is the product of numbers in the same subset. Unlike the sum–sum ratio problem, which corresponds to the assortment problem under the multinomial logit model, the sum–product ratio problem is generally NP-complete. We develop a fully polynomial-time approximation scheme and discuss several potential applications and useful extensions.

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1. Introduction

In this paper, we formulate the sum–product ratio problem and show that it has several potential applications. The sum–product ratio problem arises in the consideration set formation, under which a consumer chooses a set of products to evaluate simultaneously such that the highest utility net of search cost is maximized. It can also model some joint project assignment problems, e.g., certain projects are considered successful only if all the teams that get involved succeed.

To the best of our knowledge, we are the first to study the sum–product ratio problem. The related sum–sum ratio problem has been well studied as it is the corresponding assortment problem under the popular multinomial logit (MNL) model.

1.1. Sum–sum ratio problem

Modeling choice behavior of consumers who face multiple substitutable products has been an active research area for several decades. Perhaps one of the widely used and adopted models for consumer choice is the MNL choice model. Since it was first proposed by McFadden [12], the MNL model has received significant attention from researchers in economics, marketing, transportation and operations management communities. For thorough discussions of the MNL model, we refer the readers to Ben-Akiva and Lerman [2] and Anderson et al. [1].

Under the MNL model, the assortment problem is to choose an offer set S such that the total revenue is maximized. The problem can be formulated as follows:

$$\max_{S\subseteq\mathcal{N}} \frac{\sum_{i\in S} p_i u_i}{u_0+\sum_{i\in S} u_i}, \tag{1}$$

where p_i is the price, u_i represents the attractiveness of product i, (0 corresponds to the no-purchase or outside option,) and $\mathcal{N} := \{1, 2, \ldots, N\}$ denotes the set of all available products. In the assortment optimization literature, it has been shown that the assortment planning problems under the MNL model can be solved efficiently (see, e.g., Talluri and van Ryzin [15] and Gallego et al. [5]). The optimal assortment without a constraint has a nice structure: the revenue-ordered assortment is optimal. The readers are referred to Rusmevichientong et al. [13], Wang [16] and Besbes and Saure [3] for extensive discussions on the constrained assortment problems. In particular, Desir and Goyal [4] propose an explicit fully polynomial-time approximation scheme for the assortment problem under the MNL model and its variants. Li and Huh [10], Gallego and Wang [6], Li et al. [11] and Huh and Li [9] consider the assortment planning and pricing under the multi-stage nested logit choice models.

1.2. Sum–product ratio problem

In this paper, we study the *sum–product ratio* problem: Given N pairs of positive reals $(a_i, b_i)_{i=1}^N$, choose a subset $S \subseteq \mathcal{N} := \{1, 2, \ldots, N\}$ to maximize the sum–product ratio. The problem is

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formulated as follows:

$$\max_{S\subseteq\mathcal{N}} R(S) := \frac{\sum_{i\in S} a_i}{\prod_{i\in S} b_i}. \tag{2}$$

Without loss of generality, we assume $b_i > 1$ for each $i \in \mathcal{N}$; otherwise, item i is always included in the optimal subset. Similarly, problem (1) can be referred to as the *sum–sum ratio* problem, generally expressed by $\max_{S\subseteq\mathcal{N}} \sum_{i\in S} a_i \big/ \sum_{i\in S} b_i$.

To the best of our knowledge, this paper is the first one to formally investigate the complexity of the sum–product ratio problem and to develop efficient approximation algorithms for the sum–product ratio problem. We prove that the sum–product ratio problem is generally NP-complete by a reduction from the Partition problem, which is known NP-hard. Then, we propose a fully polynomial-time approximation scheme. This is the best possible approximation we can expect for an NP-hard problem. Inspired by Desir and Goyal [4], we discretize the parameters and use dynamic programming to find a subset such that the numerator and the denominator are close to the desired values. After repeating this procedure for all possible combinations of numerators and denominators, we can find a subset such that its objective value is close to the optimal value. We also briefly discuss several useful extensions and point out that the algorithms can possibly solve problems in higher dimensions, if the problems can be separated.

2. Applications of sum–product ratio problem

The sum–product ratio problem may arise naturally in practice. We will next discuss several important applications, e.g., the consideration set formation and joint project assignment problems.

2.1. Consideration set formation

When a consumer purchases a product, she often faces multiple options. Consumer behavior research (e.g., Hauser [8] and Wang and Sahin [17]) finds that consumers often use the two-stage choice policy: the consumer first chooses a subset of products to form her consideration set; then after searching all items in her consideration set, she chooses the one with the highest utility.

Suppose that there are N items available. The utility of each item i can be expressed as follows

$$U_i = u_i + \xi_i, \quad \forall i \in \mathcal{N},$$

where u_i represents the deterministic part of utility and ξ_i denotes its random part for item i. In the literature, it is common to assume that ξ_i follows the independent and identically distributed (i.i.d.) Gumbel distribution (see, e.g., McFadden [12]), i.e., $\Pr(\xi_i \le x) = \exp\big(-\exp(-(x/\mu+\gamma))\big)$, where $\gamma \approx 0.5772$ is the Euler's constant. Note that the mean is $E[\xi_i] = 0$ and the standard deviation is equal to $\mu\pi/\sqrt{6}$. The random term is unknown to the consumer, but it can be resolved after a costly search. The search cost may be heterogeneous for different items because the information of certain items may be more or less accessible than others. Let w_i denote the search cost for item i.

The consumer forms her consideration set S to maximize her expected utility net of the total search cost. The problem can be formulated as follows:

$$\max_{S\subseteq\mathcal{N}} E[Y_S] - \sum_{i\in S} w_i, \tag{3}$$

where $Y_S := \max_{i\in S}\{u_i+\xi_i\}$ is the highest utility for all items in set S. Under the i.i.d. Gumbel distribution, Y_S also follows the Gumbel distribution as follows: $\Pr(Y_S \le y) = \exp\big(-\exp(-((y-L)/\mu + \gamma))\big)$, where $L = \mu\log\big(\sum_{i\in S}\exp(u_i/\mu)\big)$ and $\log(\cdot)$ refers to the natural logarithm; see, e.g., Anderson et al. [1]. The mean of Y_S is equal to

$$E[Y_S] = \mu\log\Big(\sum_{i\in S}\exp(u_i/\mu)\Big).$$

Then, the consideration set formation problem (3) can be simplified as follows:

$$\max_{S\subseteq\mathcal{N}} \mu\log\Big(\sum_{i\in S}\exp(u_i/\mu)\Big) - \sum_{i\in S} w_i,$$

which is equivalent to

$$\max_{S\subseteq\mathcal{N}} \frac{\sum_{i\in S}\exp(u_i/\mu)}{\prod_{i\in S}\exp(w_i/\mu)}. \tag{4}$$

Apparently, the consideration set formation problem (4) is indeed an instance of the sum–product ratio problem (2).

2.2. Joint project assignment

Multiple teams may work together on a joint project, which can be divided into multiple independent tasks. For example, to build a personal computer system, one must develop a display, storage, user interface, printer, etc. The project, e.g., the personal computer system, is considered successful only if all teams succeed, e.g., each team designs a functional component for the personal computer system. Note that teams are different by nature, so their success probabilities may be different. On the other hand, if the project succeeds, each team may generate certain amount of revenue, e.g., each team may receive certain amount of bonuses or may represent a local market. The problem is which teams should get involved in this joint project. If too few teams work on the project, the total revenue may not be significant; if too many teams are chosen, the risk may be too high because it requires that all teams succeed. The problem is referred to as the "joint project assignment" problem, which will be formally formulated below.

Suppose that there are N teams: the success probability for team i is q_i and the revenue from team i is r_i for each $i \in \mathcal{N}$ if the joint project succeeds. The problem is to choose one or multiple teams to work on the joint project in order to maximize the total expected revenue, which can be formulated as follows:

$$\max_{S\subseteq\mathcal{N}} \sum_{i\in S} r_i \cdot \prod_{i\in S} q_i. \tag{5}$$

The joint project assignment problem is also an instance of the sum–product ratio problem (2), since it can be rewritten as follows $\max_{S\subseteq\mathcal{N}} \sum_{i\in S} r_i \big/ \prod_{i\in S}(1/q_i)$.

We next establish the hardness of the sum–product ratio problem and propose an efficient approximation algorithm.

3. Hardness of sum–product ratio problem

In the seminal paper by Talluri and van Ryzin [15], they show that the revenue-ordered assortment is optimal for the assortment problem (1) under the MNL model. In other words, an *a/b-ordered* subset defined below, is optimal for the sum–sum ratio problem (1).

Definition 1. We relabel items such that a_i/b_i is decreasing in i for any $i \in \mathcal{N}$. Then, a subset $S_n := \{1, 2, \ldots, n\}$ for each $n \in \mathcal{N}$ is referred to as an a/b-ordered subset.

Under more restrictive conditions, the optimality of the *a/b-ordered subset* may still hold for the sum–product ratio problem (2).

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