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Asymptotic behavior of optimal quantities in symmetric transshipment coalitions

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ABSTRACT

This paper addresses the asymptotic behavior of optimal quantities in symmetric transshipment coalitions. First, we provide bounds for optimal quantities under general demand structure. Second, we show that if the variance of average demand diminishes as the number of newsvendor grows, the optimal quantities move toward the distribution mean after coalitions became sufficiently large. However, the limits depend on the type of newsvendors in the coalition and the magnitude of transshipment cost above a certain threshold. We also discuss the pooling anomaly in large coalitions in these settings and show that the optimal quantities always decrease (increase) to their limit if the single newsvendor's optimal quantity is above (below) mean.

1. Introduction

Transshipment is the practice of sharing common resources among supply chain agents who face uncertain market demands. Variations of the single period transshipment problems have been extensively studied in the literature (see Paterson et al. [6] for a review). One of the complex aspects of the transshipment problems is the effect of coalition size on the optimal production/order quantities. This paper provides insights on collaborative transshipment by characterizing the asymptotic behavior of optimal quantities in symmetric transshipment coalitions.

There are a number of papers in the literature which study the effect of coalition size on optimal quantities. Dong and Rudi [3] show that in symmetric situations with normally distributed demands, the optimal quantities monotonically move toward the mean. For general demand distributions, the sequences of optimal quantities are not necessarily monotone – even in case of independent distributions (an example is given in Gerchak and Mossman [4]). Zhang [8] characterizes the behavior of optimal quantities with respect to the crossing point of individual and joint cumulative distribution functions (assuming there is only one such crossing point). To the best of our knowledge, there is no prior result with respect to the asymptotic behavior of the optimal quantities. As we show in this paper, when the average demand of the coalition has a distribution whose variance diminishes with the size of the coalition, the optimal quantities asymptotically get closer to the distribution mean, however, their sequences may never converge to the mean if the transshipment cost is larger than a certain threshold. We provide the limits of optimal quantities as well as the effective thresholds of transshipment costs. We also provide bounds for the optimal quantities in situations with general demand distributions.

The non-monotonic behavior of optimal quantities with regard to coalition size is closely related to the so-called *pooling anomaly* phenomenon. Yang and Schrage [7] prove that transshipment could increase the optimal quantities of the coalition even if the optimal quantity for an individual newsvendor is above demand mean. In fact, this can occur for any i.i.d. skewed two-point distributions. Nevertheless, Bar-Lev et al. [1] show that with i.i.d. demands and free transshipment, the latter anomaly disappears as coalition becomes large enough. We extend the last observation to situations with positive transshipment costs and demand distributions whose averages have diminishing variances.

2. Symmetric transshipment problem

Consider a collection of N newsvendors. Let D_i be the random variable representing the market demand of the newsvendor $i \in N$. We assume that the marginal distributions of $D_1, \ldots, D_n$ are identical, positive and twice differentiable on their support, and have

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finite mean and variance (μ and σ respectively). The positivity assumption is made for the ease of exposition and does not have significant effect on the results. We denote the CDF of marginal distribution by F_D; no assumption is made on the correlation structure among individual demands. The market selling price, purchasing cost, and salvage value are denoted by r, c, and ν respectively ($\nu < c < r$). The newsvendors have the option to transship their otherwise surplus products to the members of the coalition after the realization of demands. To physically move one unit of product between any two newsvendors, the transportation cost t must incur. To sustain the profitability of transshipments, we assume that $t < r - \nu$. The decision variables are the production/order quantities. Symmetry of the coalition requires that the optimal quantities for all newsvendors be the same. Let x be the quantity produced/ordered by every newsvendor in the coalition. Hezarkhani and Kubiak [5] show that the expected profit of a transshipment coalition of n symmetric newsvendors can be expressed by

$$J_n(x) = n(r-c)x - nt\int_{-\infty}^{x} F_D(\xi)d\xi - (r-\nu-t)\int_{-\infty}^{x} F_{\bar{D}_n}(\xi)d\xi, \quad (1)$$

where $\bar{D}_n = \sum_{i\in N} D_i/n$ is the average demand of n newsvendors with $F_{\bar{D}_n}$ as its CDF.

3. Optimal quantities

As J_n is concave on x, the optimal quantity can be found from the first order condition. Let $g = r - c$ be the opportunity cost of a lost sale and $\tilde{g} = c - \nu$ be the mark-down loss. The optimal quantity of each newsvendor for a coalition of size n, i.e. X_n is obtained from the following implicit function:

$$R = \frac{t}{g+\tilde{g}} F_D(X_n) + \left(1 - \frac{t}{g+\tilde{g}}\right) F_{\bar{D}_n}(X_n) \quad (2)$$

where $R = g/(g+\tilde{g})$ is the critical fractile that determines the optimal quantity of an isolated newsvendor. Consider the function L defined by

$$L(x) = \frac{g}{g+\tilde{g}-t} - \frac{t}{g+\tilde{g}-t} F_D(x). \quad (3)$$

From (2), it is clear that X_n corresponds to the intersection point of $L(x)$ and $F_{\bar{D}_n}(x)$. Clearly, $L(x)$ is independent of the size of coalition. Fig. 1 illustrates an example for this interpretation. Note that although in this example $g/(g+\tilde{g}-t) > 1$ and $(g-t)/(g+\tilde{g}-t) > 0$, this is not always the case.

The following lemma states that such an intersection point always exists and is unique. Additionally, it introduces a limiting condition on optimal quantities for high transshipment costs.

Lemma 1. *For all $n \geq 1$, X_n exists and is unique. Moreover, if $t > \tilde{g}$ we have $X_n \geq L^{-1}(1)$, and if $t > g$ we have $X_n \leq L^{-1}(0)$.*

Proof. For $n \geq 1$, the $F_{\bar{D}_n}(x)$ is an increasing and onto function on the range $(0, 1)$. If $t > 0$, then $L(x)$ is also a decreasing, one-to-one, and onto function on the range $((g-t)/(g+\tilde{g}-t), g/(g+\tilde{g}-t))$. If $t = 0$, then $L(x) = R$. Since $t < g+\tilde{g}$ we have $(g-t)/(g+\tilde{g}-t) \leq R \leq g/(g+\tilde{g}-t)$. By the assumption on the parameters, we have $0 < R < 1$. Therefore, the intersection of the range of $L(x)$ and the range of $F_{\bar{D}_n}$ is non-empty. Thus there must exist a unique $X_n \in \mathbb{R}$ such that $L(X_n) = F_{\bar{D}_n}(X_n)$. Considering the range of $F_{\bar{D}_n}$, it must be that $0 < F_{\bar{D}_n}(X_n) < 1$.

Suppose $t > \tilde{g}$. This implies that $g/(g+\tilde{g}-t) > 1$. By continuity and decreasing property of $L(x)$ there must exist c such that

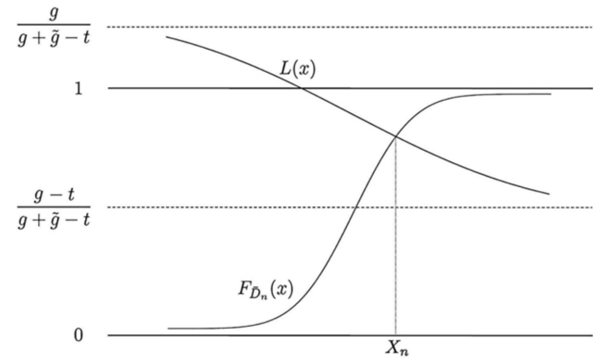

Fig. 1. Optimal quantity as the intersection point of two functions.

$L(c) > 1$. Then the Intermediate Value Theorem states that there exists the point $d \in (c, X_n)$ such that $L(d) = 1$. Therefore $L^{-1}(1)$ is well defined and we have $X_n > L^{-1}(1)$. Finally, if $t > g$ it must be that $(g-t)/(g+\tilde{g}-t) < 0$ and the same argument results in having $X_n < L^{-1}(0)$. □

The bounds in Lemma 1 are obtained considering the fact that for any $n \geq 1$, the intersection point of two function $L(x)$ and $F_{\bar{D}_n}(x)$ is located within their common range.

When the transshipment cost t is higher than certain thresholds, the quantities $L^{-1}(0)$ and $L^{-1}(1)$ play key roles in the results obtained in this paper so we provide some intuition with regard to these values. From the definition of L it follows that in case of $t > g$, the value $L^{-1}(0)$ corresponds to the quantity x where $F_D(x) = g/t$. This quantity would be optimal for a newsvendor who faces an opportunity cost of g and mark-down loss of $t-g$. Note that $t-g$ is the cost of transshipping out one unit to avoid lost sale penalty at another newsvendor facing shortage. This modified mark-down loss which takes into account the option for transshipment is in fact lower than the original mark-down loss. Thus $L^{-1}(0)$ can be understood as the optimal quantity for a newsvendor who is confident that its surplus can be sold by other newsvendors. In case of $t > \tilde{g}$, the value $L^{-1}(1)$ corresponds to the quantity x where $F_D(x) = (t-\tilde{g})/t$. This quantity would be optimal for a newsvendor who faces a lost sale opportunity cost of $t-\tilde{g}$ and mark-down loss of $\tilde{g}$. Note that $t-\tilde{g}$ is the cost of receiving one extra unit via transshipment instead of marking down that unit at another newsvendor's location. This modified opportunity cost which takes into account the option for transshipment is in fact lower than the original opportunity cost. Therefore, $L^{-1}(1)$ can be interpreted as the optimal quantity for a newsvendor who is confident that its shortages can be satisfied by receiving transshipped units from other newsvendors.

As the above result indicates, the transshipment cost is a decisive factor in the behavior of optimal quantities in transshipment coalitions. While the transshipment cost infinitely close to $g+\tilde{g}$ turns the coalition into a collection of isolated newsvendors, i.e. $X_n = X_1 = F_D^{-1}(R)$, the transshipment cost infinitely close to 0 turns the coalition into a singular newsvendor with aggregated demand, i.e. $X_n = F_{\bar{D}_n}^{-1}(R)$. For the intermediate transshipment costs, $0 < t < g+\tilde{g}$, the optimal quantities fall somewhere between the two extremes:

Lemma 2. *We have* $\min\{F_{\bar{D}_n}^{-1}(R), F_D^{-1}(R)\} \leq X_n \leq \max\{F_{\bar{D}_n}^{-1}(R), F_D^{-1}(R)\}$.

Proof. From (2) it is evident that R is a convex combination of the points $F_D(X_n)$ and $F_{\bar{D}_n}(X_n)$. If $F_D(X_n) > R$ and $F_{\bar{D}_n}(X_n) > R$, or $F_D(X_n) < R$ and $F_{\bar{D}_n}(X_n) < R$ then no such combinations can be equal to R. It is straightforward to check that all other combinations are reflected in the statement of this lemma. □