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A note on optimal risk sharing on L^p spaces

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ABSTRACT

We study Pareto optimality and optimal risk sharing in terms of convex risk measures on L^p-spaces and provide a characterization result for Pareto optimality of solutions. In comparison to similar approaches that study this problem on L^∞ this setting introduces more flexibility in terms of the underlying model space. Furthermore, in our setting agents can incorporate different risk measures where some of them reflect their own preferences and others reflect requirements from regulators.

1. Introduction

The problem of optimal risk sharing between two or more agents has been studied in several papers. The problem formulation is very general and allows one to study interactions of agents with different preferences towards risk in various contexts. It started with the early works of [7,2] with applications to insurance problems and resulted in several recently published papers like [1,10,17,19]. It has applications in different areas of finance, particularly the areas of actuarial science and portfolio theory are of great importance. The problem formulation is the following: Consider m agents, each with an initial random endowment. These initial endowments add up to an aggregate risky position. Now the optimal risk sharing problem is to find an optimal allocation of this aggregate position such that the allocated risk is acceptable to each agent. Initial endowments of the agents can represent initial endowments in a stock market but equally interpretations like randomly varying water endowments or nation's quota in producing diverse pollutants are possible, see [1]. Optimality in this context in general stands for Pareto optimality. This means that there is no other allocation such that all agents are better off according to their attitude towards risk and at least one agent of them is strictly better off. The attitude towards risk of each agent is usually represented by von Neumann–Morgenstern expected utility. Existence results and characterizations of optimal risk exchanges in this framework were obtained in various papers, see [15] for an overview. In more recent works, which study the problem from a financial risk perspective, coherent or convex risk measures have taken the role of the individual risk preference functional of each agent, see for instance [19,5]. As demonstrated in [4] the characterization of Pareto optimal allocations in this framework reduces to the calculation of the inf-convolution of convex risk measures. The inf-convolution of convex risk measures was further studied in [5] and became the basis of studying the optimal risk sharing problem in [19], where a subdifferential characterization of Pareto optimality was derived for the optimal risk sharing problem on L^∞.

In our approach we will study the coalitional risk measure (introduced in [17,16] in terms of general deviation measures) which will be based on convex risk measures. This auxiliary functional will be the key to our characterization result for the proposed risk sharing problem and for Pareto optimality of the solutions. At first we will provide an existence result for solutions to our optimal risk sharing problem on L^p, $1 \leq p \leq \infty$. Then we will study the structural properties of the coalitional risk measure. Based on the corresponding properties of the underlying risk measures we will show that it is monotone, convex and cash-subadditive and thus obtains a dual representation according to the results in [9]. Finally we will derive a characterization result for Pareto optimal solutions to this problem on L^p, $1 \leq p \leq \infty$. Thus, the main contribution of this work is, on one hand to provide an extension of the characterization result of Pareto optimality in conjunction with convex risk measures from L^∞ to L^p spaces and,

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on the other hand to introduce different techniques for the proofs of this result (instead of the inf-convolution of risk measures we will use the coalitional risk measure).

2. Notation and important properties of convex risk measures

A risky position will be modeled as a real valued random variable X from L^p, $1 \leq p \leq \infty$, on a non-atomic probability space $(\Omega, \mathscr{F}, \mathbb{P})$. As usual, we identify two random variables if they agree $\mathbb{P}$-a.s. Inequalities between random variables are understood in the $\mathbb{P}$-a.s. sense. We will introduce for each agent i her individual risk measure $\rho_i : L^p \to \mathbb{R} \cup \{\infty\}$, $i = 1, \dots, m$. This risk measure will represent the agents individual attitude towards risk. We will work with convex and coherent risk measures. Convex risk measures were introduced and studied in [13,14] based on the following properties.

(R1) $\rho(0) = 0$ and $\rho(X + r) = \rho(X) - r$
for any $X \in L^p$ and any $r \in \mathbb{R}$,
(R2) $\rho(X) \leq \rho(Y)$ for $X \geq Y$, $X, Y \in L^p$,
(R3) $\rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y)$,
for $0 \leq \lambda \leq 1$ and any $X, Y \in L^p$.

A convex risk measure with the property

(R4) $\rho(\lambda X) = \lambda\rho(X)$ for any $X \in L^p$ and $\lambda \in \mathbb{R}_+$,

is called coherent risk measure. Coherent risk measures were introduced and studied in [3]. Further properties like comonotonicity of random variables and law invariance, the Fatou and the Lebesgue property of risk measures will be needed for our main results. Law invariance of a functional $\rho : L^p \to \mathbb{R} \cup \{\infty\}$ is formulated as follows. ρ is law-invariant, if $\rho(X) = \rho(Y)$ whenever $X, Y \in L^p$ have the same probability law. A collection $(Y_1, \dots, Y_m) \in (L^p)^m$ of m real-valued random variables on $(\Omega, \mathscr{F}, \mathbb{P})$ is comonotone if for every $(i, j) \in \{1, \dots, m\}^2$

$$(Y_i(\omega') - Y_i(\omega))(Y_j(\omega') - Y_j(\omega)) \geq 0,$$

for $\mathbb{P} \otimes \mathbb{P}$-a.e. $(\omega', \omega) \in \Omega^2$.

A functional $\rho : L^p \to \mathbb{R} \cup \{\infty\}$ satisfies the Lebesgue property if for every uniformly bounded sequence $(X_n)_{n=1}^\infty$ tending a.s. to $X \in L^p$ we have $\rho(X) = \lim_{n\to\infty} \rho(X_n)$. ρ satisfies the Fatou property if we replace the equality in the Lebesgue property with $\rho(X) \leq \liminf_{n\to\infty} \rho(X_n)$. Additionally the concept of convex orders will be needed in the following. Let $X, Y \in L^p$, then X dominates Y in the convex order if and only if $\mathbb{E}(\varphi(X)) \geq \mathbb{E}(\varphi(Y))$ for any convex function $\varphi : \mathbb{R} \to \mathbb{R}$.

Furthermore the subdifferential of a convex function will play an important role. Here we will follow the notation from [23]. A linear functional $l : L^p \to \mathbb{R}$ is called algebraic subgradient of a convex functional ρ on L^p at $Z \in \operatorname{dom} \rho$ if

$$\rho(X') \geq \rho(Z) + l(X' - Z) \quad \forall X' \in L^p. \tag{1}$$

We will denote the dual space of $L^p(\Omega, \mathscr{F}, \mathbb{P})$, $1 \leq p \leq \infty$ by $L^p(\Omega, \mathscr{F}, \mathbb{P})^*$. The dual space of $L^\infty(\Omega, \mathscr{F}, \mathbb{P})$ can be identified with $ba(\Omega, \mathscr{F}, \mathbb{P})$, the space of bounded finitely additive signed measures μ on $(\Omega, \mathscr{F})$ such that $\mathbb{P}(A) = 0$ implies $\mu(A) = 0$. The dual space of L^1 can be identified with L^∞ and the dual space of L^p for $1 < p < \infty$ is L^q, where $1/p + 1/q = 1$. If l is from the dual space of L^p, $1 \leq p \leq \infty$ and satisfies (1) we will call it subgradient of ρ. The set of all subgradients of ρ at Z will be denoted by $\partial\rho(Z)$ and we will call $\partial\rho(Z)$ the subdifferential of ρ at Z. It is said that ρ is subdifferentiable at Z if $\partial\rho(Z)$ is nonempty. Moreover, we will need the Fenchel conjugate of a proper convex risk measure on L^p which is a function α from the dual space $L^p(\Omega, \mathscr{F}, \mathbb{P})^*$ of $L^p(\Omega, \mathscr{F}, \mathbb{P})$ to $\mathbb{R} \cup \{\infty\}$ that is defined by

$$\alpha(l) = \sup_{X \in L^p(\Omega, \mathscr{F}, \mathbb{P})} \{l(X) - \rho(X)\}$$

Note that property (R1) implies that convex risk measures are proper. For convex risk measures on L^∞ this implies finiteness and continuity on L^∞. Furthermore we know that any finite convex risk measure on L^p, $1 \leq p \leq \infty$ is continuous and subdifferentiable on L^p, see Proposition 3.1 in [23] or Corollary 2.3 in [20].

Remark 2.1. In the main part of the paper we will work with law-invariant convex risk measures. In this regard, note that Jouini et al. [18] have shown for convex functionals on L^∞ which are law-invariant and lower semicontinuous with respect to the topology induced by $\|\cdot\|_\infty$ that these are lower semicontinuous with respect to the $\sigma(L^\infty, L^1)$-topology (see Theorem 2.2 in [18]). This implies for finite law-invariant convex risk measures on L^∞ a dual representation

$$\rho(X) = \sup_{Z \in L^1} \{\mathbb{E}[ZX] - \alpha(Z)\} \tag{2}$$

where the supremum is formed over L^1 instead of ba. If a finite law-invariant convex risk measure (which automatically satisfies the Fatou property) additionally satisfies the stronger Lebesgue property the supremum in (2) is attained for every $X \in L^\infty$ (see Theorem 5.2 in [18]) and it follows from (2) and the characterization of the subgradient of a convex risk measure ρ at X, namely

$$Z \in \partial\rho(X) \Leftrightarrow \rho(X) = \mathbb{E}[ZX] - \alpha(Z),$$

that the subgradients of ρ at X are in L^1, i.e. $\partial\rho(X) \subset L^1$. We will use this fact in Corollaries 4.3 and 4.9. With respect to the Lebesgue property of finite convex risk measures on L^p, $1 \leq p < \infty$, note that these risk measures are automatically Fatou and Lebesgue continuous, see Theorem 3.1 in [20].

3. The risk sharing problem

We are interested in the following problem. Suppose there are m agents who view risk differently and who are bound to different multiple regulatory requirements. Consequently each agent i is equipped with multiple individual risk measures

$$\rho_i^1, \dots, \rho_i^{r_i}$$

where some of them may reflect her own preferences and other are regulatory requirements. Following the motivation in [16] the agents may find different aspects of an asset to be attractive and thus they may decide to form a joint portfolio in which the share of investor i is preferred over the optimal portfolio that investor i can form alone. Then the question arises, how to divide the future payoff of cooperative portfolio X among the agents. Thus we consider divisions $Y = (Y_1, \dots, Y_m)$, $Y_i \in L^p$, $i = 1, \dots, m$ of X such that $\sum_{i=1}^m Y_i = X$. We will denote the set of these divisions by

$$\mathscr{A}(X) = \left\{ Y = (Y_1, \dots, Y_m) \in (L^p)^m \,\middle|\, \sum_{i=1}^m Y_i = X \right\}$$

and by $\mathscr{C}(X) \subset \mathscr{A}(X)$ we will denote the set of random vectors from $\mathscr{A}(X)$ with comonotone components.

We will assume that each investor takes into account her individual risk measures ρ_i^j by applying cautious risk measurement methods in order to meet worst case situations and chooses to use

$$\rho_i^{\max}(X) = \max\{\rho_i^1(X), \dots, \rho_i^{r_i}(X)\}$$

as her risk measure.

Note that since we consider the maximum risk measure for each agent we allow each agent to have her individual number of different risk measures. In this way agents can incorporate any number of different risk measures where some of them reflect their own