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A new rule for the problem of sharing the revenue from museum passes

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ABSTRACT

We present a new rule for the problem of sharing the revenue from museum passes. The rule allocates the revenue from each pass proportionally to the product of the admission fee and the number of total visits (with and without pass) of the museums. We provide a systematic study of the properties of the rule, in comparison with other rules in the literature.

1. Introduction

The problem of sharing the revenue from museum passes is a focal (real-life) instance of revenue sharing problems under bundled pricing. In numerous cities worldwide, there exist passes offering access to several museums, for a price below the aggregate admission fee of those museums. The problem is to share the net revenue from the sale of passes among the participating museums. The original formalization of this problem is in [7]. For a survey of contributions on this problem, the reader is referred to [5].

In a recent paper [3], we have presented two models generalizing those previous contributions to analyze *museum problems*. Our main contribution therein is to bring additional aspects (such as admission fees and the number of visits without the pass of each museum) into the analysis. In both models, which differ on their informational bases, we provide normative, as well as game-theoretical, justifications for several rules considering those aspects. The aim of this note is to introduce two new rules (one for each of the two models considered), which seem to be superior to the existing ones on several grounds. The common principle that both rules implement is to allocate the revenue among the museums proportionally to the product of the admission fee and the number of total visits (with and without pass) of the museums. The note is devoted to provide a systematic study of the properties of both rules, in comparison with other rules in the literature.

2. The benchmark model

We start considering the first model introduced in [3], which itself generalizes the seminal model introduced in [7] and studied later in [2,8,9].

A (museum) **problem** is a 6-tuple (M, N, π, K, p, v) where M is a (finite) set of **museums**, N is a (finite) set of **pass holders** whose cardinality we denote by n, $\pi \in \mathbb{R}_+$ is the **pass price**, $K \in 2^{nM}$ is the profile of (non-empty) **sets of museums visited by each pass holder**, $p \in \mathbb{R}^m_{++}$ is the profile of **admission fees**, and $v \in \mathbb{Z}^m_+ \setminus \{0\}$ is the profile of **visits without pass**. The family of all the problems so described is denoted by $\mathcal{P}$.

For each $l \in N$, let $K_l \subset M$ denote the set of museums visited by pass holder l. For each $i \in M$, let $U_i(K)$ denote the set of pass holders visiting museum i. Namely, $U_i(K) = \{j \in N : i \in K_j\}$. Finally, let $k_l = |K_l|$, for each $l \in N$, and $\nu_i = |U_i(K)|$, for each $i \in M$.

A **rule** is a mapping that associates with each problem an allocation indicating the amount each museum gets from the revenue generated by passes sold. Formally, $R : \mathcal{P} \to \mathbb{R}^m_+$ is such that, for each $(M, N, \pi, K, p, v) \in \mathcal{P}$, $\sum_{i \in M} R_i(M, N, \pi, K, p, v) = n\pi$.

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We impose from the outset that rules satisfy two basic axioms. The first one, **equal treatment of equals**, states that if two museums have the same visitors with pass, the same admission fee, and the same number of independent visits, then they should receive the same amount. Formally,

ETE: For each $(M, N, \pi, K, p, v) \in \mathcal{P}$, and each pair $i, j \in M$ such that $(U_i(K), p_i, \nu_i) = (U_j(K), p_j, \nu_j)$, $R_i(M, N, \pi, K, p, v) = R_j(M, N, \pi, K, p, v)$.

The second one, known as the **dummy** axiom, states that if nobody visits a given museum with the pass, then such a museum gets no revenue. This property has game-theoretical implications as it guarantees that the rule always selects an allocation within the core of the associated TU-game to a museum problem (e.g., [3]). Formally,

D: For each $(M, N, \pi, K, p, v) \in \mathcal{P}$, and each $i \in M$, such that $U_i(K) = \emptyset$, we have $R_i(M, N, \pi, K, p, v) = 0$.

In [3], we study several rules for this model. One of them (S^{pv}) brings independent visits (i.e., visits without the pass) into the picture. The rule is formally defined as follows. For each $(M, N, \pi, K, p, v) \in \mathcal{P}$, and $i \in M$,

$$S_i^{pv}(M, N, \pi, K, p, v) = \sum_{l \in N, i \in K_l} \frac{p_i \nu_i}{\sum_{j \in K_l} p_j \nu_j} \pi.$$

S^{pv} is subject to an important criticism articulated next. Assume that two museums i and j with the same admission fee, i.e., $(p_i = p_j)$, received the same large set of visitors with the pass (say, for instance, that $v_i = v_j = 1000$). Now, museum i had only one visitor without the pass, whereas museum j had two, i.e., $(\nu_i = 1 < 2 = \nu_j)$. In this example, it seems reasonable that museum j receives a slightly higher award than museum i. Nevertheless, S^{pv} awards museum j with twice the amount received by museum i, which seems to be excessive and unfair.

Motivated by this, we present a new rule, which is immune to such a criticism. More precisely, the **price-visits weighted rule** (**W**) allocates the revenue from each pass among the museums visited by the user of such a pass, proportionally to the product of the admission fee and the number of total visits (with and without pass) of the museums. Formally, for each $(M, N, \pi, K, p, v) \in \mathcal{P}$, and $i \in M$,

$$W_i(M, N, \pi, K, p, v) = \sum_{l \in N, i \in K_l} \frac{p_i(v_i + \nu_i)}{\sum_{j \in K_l} p_j(v_j + \nu_j)} \pi.$$

The price-visits weighted rule satisfies the axiom of **proportionality to visits**, which refers to the effect that the number of visits (with and without pass) should have on the outcome. More precisely, consider two museums with the only difference that one doubles the total visits of the other. In such a case, it seems natural that the revenue of the former be twice the revenue of the latter. More generally, the axiom says the following:

PV: For each $(M, N, \pi, K, p, v) \in \mathcal{P}$ and each pair $i, j \in M$ such that $U_i(K) = U_j(K)$, $p_i = p_j$ and $\nu_i \le \nu_j$, $R_j(M, N, \pi, K, p, v) = \frac{v_j+\nu_j}{v_i+\nu_i} R_i(M, N, \pi, K, p, v)$.

As shown in Theorem 1, W satisfies this axiom, whereas S^{pv} does not. Conversely, S^{pv} satisfies the following axiom (which we name **proportionality to independent visits**), whereas W does not. The axiom extends the argument outlined in the example presented above, which illustrated the criticism against S^{pv}.

PIV: For each $(M, N, \pi, K, p, v) \in \mathcal{P}$ and each pair $i, j \in M$ such that $U_i(K) = U_j(K)$, $p_i = p_j$ and $\nu_i \le \nu_j$, $R_j(M, N, \pi, K, p, v) = \frac{\nu_j}{\nu_i} R_i(M, N, \pi, K, p, v)$.

An alternative to the previous axioms is **marginality**, which states that, among two museums only differing in the number of independent visits, the relative increase on the revenue of one museum over the other should be the relative increase of the visits of the former museum with respect to the total number of visits of the latter. Formally,

M: For each $(M, N, \pi, K, p, v) \in \mathcal{P}$ and each pair $i, j \in M$ such that $U_i(K) = U_j(K)$, $p_i = p_j$ and $\nu_i \le \nu_j$,

$$\frac{R_j(M, N, \pi, K, p, v) - R_i(M, N, \pi, K, p, v)}{R_i(M, N, \pi, K, p, v)} = \frac{\nu_j - \nu_i}{v_i + \nu_i}.$$

Table 1
Behavior of rules W and S^{pv}.

Rules	Axioms				
	ETE	*D*	*PV*	*PIV*	*M*
W	YES	YES	YES	NO	YES
S^{pv}	YES	YES	NO	YES	NO

Table 1 summarizes the behavior of both rules with respect to the previous axioms, whereas the result proves them formally.

Theorem 1. *The following statements hold:*

- *W satisfies equal treatment of equals, dummy, proportionality to visits and marginality, whereas it does not satisfy proportionality to independent visits.*
- *S^{pv} satisfies equal treatment of equals, dummy and proportionality to independent visits, whereas it does not satisfy proportionality to visits and marginality.*

Proof. It is obvious that W satisfies **equal treatment of equals** and **dummy**. We prove that W satisfies **proportionality to visits**. Formally, let $(M, N, \pi, K, p, v) \in \mathcal{P}$ and $i, j \in M$ be such that $U_i(K) = U_j(K)$, $p_i = p_j$ and $\nu_i \le \nu_j$. Then,

$$W_j(M, N, \pi, K, p, v) = \sum_{l \in N, j \in K_l} \frac{p_j(v_j + \nu_j)}{\sum_{j' \in K_l} p_{j'}(v_{j'} + \nu_{j'})} \pi = \sum_{l \in N, j \in K_l} \frac{p_j(v_i + \nu_i)\frac{v_j+\nu_j}{v_i+\nu_i}}{\sum_{j' \in K_l} p_{j'}(v_{j'} + \nu_{j'})} \pi.$$

As $U_i(K) = U_j(K)$, $j \in K_l$ if and only if $i \in K_l$. Besides, $p_i = p_j$. Thus,

$$W_j(M, N, \pi, K, p, v) = \frac{v_j + \nu_j}{v_i + \nu_i} \sum_{l \in N, i \in K_l} \frac{p_i(v_i + \nu_i)}{\sum_{j' \in K_l} p_{j'}(v_{j'} + \nu_{j'})} \pi = \frac{v_j + \nu_j}{v_i + \nu_i} W_i(M, N, \pi, K, p, v).$$

We now prove that W satisfies **marginality**. Formally, let $(M, N, \pi, K, p, v) \in \mathcal{P}$ and $i, j \in M$ be such that $U_i(K) = U_j(K)$, $p_i = p_j$ and $\nu_i \le \nu_j$. Then,

$$\frac{W_j(M, N, \pi, K, p, v) - W_i(M, N, \pi, K, p, v)}{W_i(M, N, \pi, K, p, v)} = \frac{\sum_{l \in N, j \in K_l} \frac{p_j(v_j+\nu_j)}{\sum_{j' \in K_l} p_{j'}(v_{j'}+\nu_{j'})}\pi - \sum_{l \in N, i \in K_l} \frac{p_i(v_i+\nu_i)}{\sum_{j' \in K_l} p_{j'}(v_{j'}+\nu_{j'})}\pi}{\sum_{l \in N, i \in K_l} \frac{p_i(v_i+\nu_i)}{\sum_{j' \in K_l} p_{j'}(v_{j'}+\nu_{j'})}\pi}$$

$$= \frac{\sum_{l \in N, j \in K_l} \frac{p_j(\nu_j-\nu_i)}{\sum_{j' \in K_l} p_{j'}(v_{j'}+\nu_{j'})}}{\sum_{l \in N, i \in K_l} \frac{p_i(v_i+\nu_i)}{\sum_{j' \in K_l} p_{j'}(v_{j'}+\nu_{j'})}} = \frac{p_j(\nu_j - \nu_i) \sum_{l \in N, j \in K_l} \frac{1}{\sum_{j' \in K_l} p_{j'}(v_{j'}+\nu_{j'})}}{p_i(v_i + \nu_i) \sum_{l \in N, i \in K_l} \frac{1}{\sum_{j' \in K_l} p_{j'}(v_{j'}+\nu_{j'})}} = \frac{\nu_j - \nu_i}{v_i + \nu_i}$$