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Equivalence nucleolus for coalitional games with externalities

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ABSTRACT

The objective of this paper is to develop a solution concept for stability of coalitional games with externalities. The existing solution concepts for this class of coalitional games can be empty. Using the partition function form representation, we propose a new solution concept called *equivalence nucleolus*, which is shown to be unique and always non-empty.

1. Introduction

In coalitional games, credible communication among players takes place. Players are allowed to form coalitions and make binding agreements on how to share the payoffs of the coalitions. When a game is played under such settings, there are two fundamental questions to be answered: (i) *Which coalitions will form*? (ii) *How to divide the payoffs*? In coalitional game theory, *stability* is the answer often associated with the question — *which coalitions will form*? An allocation is said to be stable when no coalition has incentive to deviate. Stability is a necessary but not a sufficient condition for coalition formation. An unstable coalition never forms, but a stable coalition need not necessarily form [15]. In this paper we restrict ourselves to addressing the issue of stability of a solution concept.

In coalitional game theory, a game is predominantly represented in characteristic function form, which inherently assumes that the payoff to a coalition is independent of the structure of the other coalitions that exist in the game. There are many solution concepts to analyze the stability of characteristic function games e.g. the core, the nucleolus, the bargaining set and the kernel. However, there is a limitation to these solution concepts. They cannot be applied when payoff to a coalition also depends on the non-members existing in the game [1]. This is a situation of externality. In coalitional game theory, externality is defined as a situation when payoff of a coalition, not only depends on the members of the coalition, but also depends on how the other coalitions in the game structure themselves. For example, alliance formation in airlines industry involves externalities, where the market share captured by a player not only depends on its own alliance but also depends on whether other players in the market form some alliance or compete with each other.

As mentioned earlier, the characteristic function games cannot be used to model many socio-economic settings that involve externalities. To capture such situations, the *partition function form* [14] is used, in which each coalition is assigned a payoff depending on the coalition itself as well as the entire coalition structure. Many extensions of the *core* of characteristic function games have been proposed in the literature to analyze the stability of coalitional games with externalities which are represented by partition function form games. In such games, a coalition can have multiple values depending on how the outside players partition themselves. Hence, while testing the coalitional deviations in the core, certain behavioral assumptions like optimism and pessimism about the reaction of outside players are made. Such assumptions lead to different outcomes and lack of coherence in the existing literature. This issue is well taken in the solution concept γ-core [4] which is based on the individual's best strategy of residual players, and further improved in the solution concepts r-core [6] and the recursive core [8] which allow arbitrary reactions [1]. Nevertheless, it is well known that the core of coalitional games in the presence of externalities can be empty [8,5]. A discussion on axiomatic foundation of these solution concepts is found in [3]. Another important issue which is usually mentioned in connection with the solution concepts is farsightedness of players for which the equilibrium binding agreement [12] is perhaps the most notable solution concept. This solution concept, which involves counter-objection to an objection with a high degree of refinement, is immune to any credible deviations.

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Along with various extensions of the core, other classical solution concepts like the bargaining set [2] and the nucleolus [13], which are always nonempty for characteristic function games, have also begun to receive attention for the extension to the games with externalities [9]. This is the basic premise of our work. In our model, we introduce a concept, called *bargaining power* and define a payoff division rule, called *equality of satisfaction values* to obtain stable outcome. The payoff division rule is motivated from the egalitarian solution of a two person bargaining problem [10], which is guided by the *equal gain principle*. Under this division rule, the players within a coalition bargain for equal satisfaction values. We show that the division rule is an equivalence relation. Using the fundamental theorem of equivalence relation, we prove that the stable outcome is always nonempty. It is also shown that any division rule, if it is an equivalence relation, does not always give a non-empty stable outcome. We call our model *equivalence nucleolus*.

2. Preliminaries

A non-empty finite set of players N is given. A coalition C is a subset of N. Structuring of N into a set of disjoint coalitions is called a partition of N, denoted by P. An embedded coalition over N is a pair of the form (C, P) where $C \in P$. E_N is the set of all embedded coalitions over N. A *characteristic function* $v : 2^N \mapsto \mathbb{R}$, associates with each coalition $C \subseteq N$, a real valued payoff $v(C)$ that the coalition's members can distribute among themselves. Also $v(\emptyset) = 0$. A *partition function* $u : E_N \mapsto \mathbb{R}$ is a mapping that assigns a real number $u(C, P)$ to each embedded coalition (C, P). A payoff vector $x \in \mathbb{R}^{\mathbb{N}}$ is an n-dimensional real vector whose element x_i is the amount received by player i, when dividing the payoff generated by N among its members. Externality on C due to change in partition from P_1 to P_2, where $C \in P_1, P_2$, is equal to $u(C, P_1) - u(C, P_2)$. A *characteristic function game* is represented as a pair $(N, v(C))$ and a *partition function game* is represented as a pair $(N, u(C, P))$.

2.1. Key definitions

If $P = \{C_1, C_2, \ldots, C_r, \phi\}$ and $Q = \{B_1, B_2, \ldots, B_s, \phi\}$ are two partitions and $\forall i = 1, 2, \ldots, s \exists k \in \{1, 2, \ldots, r\}$ such that $C_k \neq \phi$, then Q is called the *refinement* of partition P, if $B_i \subseteq C_k \in P$. For any partition P of N and a coalition $S \notin P$, *residual partition* of P with respect to coalition S, denoted by P'_S, is given by $P'_S = \{C | \exists B \in P \text{ such that } C = B - S\} \cup \{S\}$. A payoff vector $x \in \mathbb{R}^{\mathbb{N}}$ to the players of a game $(N, u(C, P))$ is *admissible* to partition P, if $\forall C \in P, \sum_{i \in C} x_i = x_C = u(C, P)$. A payoff vector $x \in \mathbb{R}^n$ is *individually rational*, if $\forall i \in N, x_i \geq \min_{P' \in \Pi} u(\{i\}, P')$ where Π is a set of all partitions in the form of $P_{N-\{i\}} \cup \{i\}$ and $P_{N-\{i\}}$ denotes partition of the set $N - \{i\}$. For the suitability to our model, we define individual rationality instead of using an already existing and a more convenient term, called participation rationality [8] which is defined as $x_i \geq 0 \forall i \in N$. *Imputation set* $I(x)$ is a set of payment vectors $x \in \mathbb{R}^n$ of a game $(N, u(C, P))$, where x is admissible and individually rational. A *payoff configuration* to a game $(N, u(C, P))$, is a pair (P, x) where P is a partition of N and x is a payoff vector admissible to P. An *outcome* of a game $(N, u(C, P))$ is a payoff configuration (P, x) to that game where x is in the imputation set [7,9].

3. The solution concept

3.1. Bargaining power

Consider a partition function game $(N, u(C, P))$ and a payoff configuration (P, x) associated with it. A deviation by a set of players $S \notin P$ leads to the residual partition P'_S. If $\sum_{i \in S} x_i < u(S, P'_S)$, then the players constituting S have incentive to deviate from their affiliations in P and form P'_S. P'_S puts externalities on residual coalitions, thereby changing their potential payoffs. This provokes residual players to restructure themselves which may not be good for S. Therefore the players in S would not deviate, if there is a scope of losing due to residual players' actions. In short, a deviation is not credible, if it can be nullified or countered. We consider that credible deviation of a player reflects its influence in a game. Hence, for every $i \in N$ in a game $(N, u(C, P))$, *bargaining power* of player i, denoted by B_i, is defined as a real number which a player assigns himself as a measure of his influence in the game. It is an intrinsic value of each player in the game which means that the value does not change in the event of the player changing partitions.

Definition 1 (*Objection*). Let (P, x) be a payoff configuration to a game $(N, u(C, P))$. An objection of a coalition $S \subseteq N$ against (P, x) is a payoff configuration (P'_S, y), where P'_S is the residual partition of P with respect to S such that $S \notin P$, $S \in P'_S$ and $y \in \mathbb{R}^n$ is a payoff vector admissible to P'_S. Therefore $\forall C \in P'_S, \sum_{i \in C} y_i = u(C, P'_S)$. Also $\forall i \in S, y_i \geq x_i$ and $\exists i \in S$ such that $y_i > x_i$.

Definition 2 (*Counter-objection*). Let (P, x) be a payoff configuration to a game $(N, u(C, P))$ and (P'_S, y) be an objection of S against (P, x). A counter-objection of a coalition $T \subset N - S$ against (P'_S, y) is a payoff configuration (R, z), where R is the residual partition of P'_S with respect to T such that $T \notin P'_S$, $T \in R$ and $z \in \mathbb{R}^n$ is a payoff vector admissible to R. Therefore $\forall C \in R, \sum_{i \in C} z_i = u(C, R)$. Also $\forall j \in T, z_j \geq y_j$ and $\exists i \in S$ such that $z_i < x_i$.

It is important to mention here, that an objection is not defined on a single player, but on a set of players which may involve players from different coalitions. Therefore it is not necessary that a partition with each player forming a coalition by himself will always be a configuration without objection. The intuition behind these definitions is explained as follows. A partition P is given. In objection, some players from different coalitions find that if they come out of their coalitions and form a new coalition (say S) then some of them will be strictly better off while others will be at least indifferent. So they propose a new partition P'_S in which the coalition S is realized. In counter-objection, a coalition (say T) consisting of some of the remaining players may come up with another partition R in which no member of the coalition T loses, but at least one member of the coalition S strictly loses. Due to this, the losing members of S will not come out of their initial affiliations in P and coalition S will never be realized.

The steps given below are followed to compute the bargaining power of a player:

1. **Step 1:** Choose a payoff configuration (P, x) arbitrarily. Check for any objection (P'_S, y) to it according to Definition 1. If there is no objection, go to step 3.
2. **Step 2:** If (P'_S, y) is an objection to (P, x), check if there exists a counter-objection (R, z) to it according to Definition 2. If a counter-objection exists, neglect the objection as it is not a credible objection.
3. **Step 3:** Repeat the above steps for all the given partitions and associated payoff vectors, unless all the possibilities are exhausted and there is no credible objection.
4. **Step 4:** If (P, x) is the only payoff configuration with no credible objection, then the lower bound of each element of the payoff vector x gives the bargaining power of the corresponding player.

There are two possible issues with the computation of bargaining power in Step 4.