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On packing and covering polyhedra in infinite dimensions

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ABSTRACT

We consider the natural generalizations of packing and covering polyhedra in infinite dimensions, and study issues related to duality and integrality of extreme points for these sets. Using appropriate finite truncations we give conditions under which complementary slackness holds for primal/dual pairs of the infinite linear programming problems associated with infinite packing and covering polyhedra. We also give conditions under which the extreme points are integral. We illustrate an application of our results on an infinite-horizon lot-sizing problem.

1. Introduction

Packing and covering polyhedra have been useful tools in optimization to model a wide variety of decision problems involving resource allocation or demand satisfaction. This has certainly been the case in combinatorial optimization, where many fundamental results involve packing and covering polyhedra, particularly in the study of primal/dual pairs of such polyhedra and the integrality of their extreme points. One salient example is König's Theorem, which establishes the duality relationship between the matching and vertex cover polyhedra of a bipartite graph, two polyhedra known to have integral extreme points.

In the decades since, packing and covering polyhedra have been routinely applied in finite-dimensional combinatorial optimization. For example, Fulkerson, who called them anti-blocking and blocking polyhedra respectively [15–17], used them to derive polarity relations in various combinatorial settings and applied them to several fundamental problems in graph theory and discrete optimization, e.g. the study of perfect graphs and the derivation or re-derivation of duality results such as the max-flow min-cut theorem.

In this paper we study the generalization of packing and covering polyhedra to problems in infinite dimensions. We first consider strong duality results for the infinite linear programming problems (LP) associated with such polyhedra. Unfortunately, the LPs associated with packing and covering polyhedra often have objective functions that are infinite-valued over some or all of the feasible region. An example where such infinite-valued objective functions naturally appear is infinite-horizon planning problems. A complication that arises from infinite objective values is that some solutions attaining an infinite objective value may be more desirable than others but the objective value does not distinguish this. A common approach to resolve this is to introduce discount factors or running averages to force the objective function to be finite-valued and allow meaningful comparison between solutions. However, these approaches introduce strong biases towards earlier or later periods, respectively, which may be undesirable in some situations. To overcome this issue, we consider a notion of optimality that allows for the comparison of solutions with infinite objective values. This notion extends an idea from combinatorial optimization in infinite graphs and utilizes the complementary slackness condition from strong duality to characterize optimality.

In order to implement this idea of using complementary slackness as a notion of optimality, we introduce a primal/dual pairing between infinite packing and covering linear programs that is easy to construct formally, without using algebraic or topological duality. In general, this is only helpful if the proposed pairing is indeed a primal/dual pair in some standard sense. A contribution of our paper is to show that for the case of packing/covering linear programs, complementary slackness makes sense for our proposed

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primal/dual pairing. Specifically, one can define optimality by means of complementary slackness of a pair of solutions and show that such a pair of solutions exists under reasonable assumptions (Theorem 11). As an application of these tools, we derive properties for the optimal policies of an infinite horizon lot-sizing problem with an undiscounted and infinite-valued objective function.

Our strong duality results are limited to infinite packing and covering polyhedra defined by linear constraints with only a finite number of non-zero coefficients. Extending this result to more general packing and covering polyhedra will likely require more advanced techniques; one possibility is to apply discrete techniques from infinite graph theory. For this reason we consider the integrality of extreme points in infinite packing and covering polyhedra as a first step towards extending these results.

The remainder of this paper is organized as follows. In Section 2 we introduce notation and definitions, and review some previous work. In Section 3 we show the strong duality result, while in Section 4 we show an application of this result to an infinite horizon lot-sizing problem. Finally, in Section 5 we give the result on integrality of extreme points.

2. Definitions and previous work

For arbitrary sets $\mathbb{W}$ and L, we let $\mathbb{W}^L$ denote all functions from L to $\mathbb{W}$. This set can also be thought as tuples of elements in $\mathbb{W}$ indexed by L or as the Cartesian product $\prod_{\ell\in L}\mathbb{W}$. Some of the proof techniques will employ the product topology of $\mathbb{W}^L$, where convergence is characterized by coordinate-wise convergence. This convergence is also known as point-wise convergence. If $\mathbb{W}$ is a compact topological space, then $\mathbb{W}^L$ is compact when endowed with the product topology by Tychonoff's theorem. We also need the following partial converse to the Krein–Milman theorem:

Theorem 1 (*Milman, Quoted From [14]*). *If K is a compact convex subset of a locally convex space and if $A \subseteq K$ is such that K is equal to the closure of the convex hull of A, then the extreme points of K are contained in the closure of A.*

We also need an appropriate definition of infinite and possibly uncountable sums. Fortunately, because we will only consider sums of non-negative numbers we can use the following straightforward definition.

Definition 2. Let I be an arbitrary and possibly uncountable index set and let $a \in \mathbb{R}^I_+$. We let $\sum_{i\in I} a_i := \sup_{S:S\subseteq I,\,|S|<\infty} \sum_{i\in S} a_i$.

We can now introduce packing and covering pairs of LPs through the following definition.

Definition 3 (*Packing–Covering Pair*). Let I, J be arbitrary and possibly uncountable index sets, $A := (a_{ij})_{i\in I, j\in J} \in \mathbb{R}^{I\times J}_+$ be a possibly doubly infinite and non-negative "matrix", and $c := (c_j)_{j\in J} \in \mathbb{R}^J_+$ and $b := (b_i)_{i\in I} \in \mathbb{R}^I_+$ be non-negative and possibly infinite "vectors". We consider the packing–covering pair of LPs given by

$$(P)\quad z^* = \inf \sum_{j\in J} c_j x_j \tag{1a}$$

$$\text{s.t.}\quad \sum_{j\in J} a_{ij} x_j \geq b_i \quad \forall\, i \in I \tag{1b}$$

$$x_j \geq 0 \quad \forall\, j \in J \tag{1c}$$

and

$$(D)\quad w^* = \sup \sum_{i\in I} b_i y_i \tag{2a}$$

$$\text{s.t.}\quad \sum_{i\in I} a_{ij} y_i \leq c_j \quad \forall\, j \in J \tag{2b}$$

$$y_i \geq 0 \quad \forall\, i \in I. \tag{2c}$$

The *covering polyhedron* defined by A and b is the feasible region of (P) given by $\mathcal{P}_\uparrow(A, b) := \{x \in \mathbb{R}^J : \text{(1b)–(1c)}\}$ and the *packing polyhedron* defined by A and c is the feasible region of (D) given by $\mathcal{P}_\downarrow(A, c) := \{y \in \mathbb{R}^I : \text{(2b)–(2c)}\}$. When A, b and c are clear from the context, we use the notation $\mathcal{P}_\uparrow$ and $\mathcal{P}_\downarrow$.

If I and J are finite, this pair of problems reduces to a traditional finite packing–covering pair. In addition, special infinite versions of this problem have been studied by several authors. [13] studies the extreme points of the packing polyhedron associated with the stable sets of an infinite perfect graph, and [19] does the same for the packing polyhedron associated with matchings and b-matchings of a bipartite infinite graph. In a series of papers, Aharoni and his co-authors study duality for several problems in infinite graphs and hypergraphs, including integer and fractional matching [3,5,8,10,9], connectivity [4,7] and flows [6]. Another stream of related work is a series of papers by Romeijn, Smith and their co-authors. These papers study extreme points [12,18] and duality [22,23] for problems with more general structure, but only with countably infinite I and J; i.e. for the case in which both $\mathcal{P}_\uparrow$ and $\mathcal{P}_\downarrow$ are contained in $\mathbb{R}^{\mathbb{N}} = \prod_{i=1}^{\infty}\mathbb{R}$. When applied to countably infinite packing–covering pairs the results in [12] imply that the extreme points of the finite projections of $\mathcal{P}_\downarrow$ converge in the product topology of $\mathbb{R}^{\mathbb{N}}$ to the closure of the extreme points of $\mathcal{P}_\downarrow$ and hence, if the extreme points of the finite projections are all integral, then the extreme points of $\mathcal{P}_\downarrow$ are also integral. The results in [22,23] imply that, under some technical conditions, strong duality holds for (P)/(D) for countable I, J.

While the previous papers are the most related to our results, infinite LP has been widely studied; we refer the reader to [11] for a longer treatment. In particular, many authors consider strong duality results for primal/dual pairs of infinite LPs in uncountable dimensions using topological or measure-theoretic techniques. An example of such a result and its application in operations research can be found in [1,2,20,21]. However, our formulation is more naturally an extension of existing research for the countable case, because our dual constructions do not rely on linear functionals (and thus are not duals in the classical sense) and because we explicitly allow objectives and constraints to evaluate to infinity.

3. LP duality

The following straightforward lemma shows that weak duality holds for (P)/(D).

Lemma 4 (*Weak Duality*). $w^* \leq z^*$

Proof. Let $\bar{x}$ and $\bar{y}$ be feasible for (P) and (D) and let $I^+ = \{i \in I : \bar{y}_i > 0\}$. Then

$$\begin{aligned}\sum_{j\in J} c_j\bar{x}_j &\geq \sum_{j\in J}\bar{x}_j\sum_{i\in I} a_{ij}\bar{y}_i = \sum_{j\in J}\bar{x}_j\sum_{i\in I^+} a_{ij}\bar{y}_i\\ &= \sum_{j\in J}\sum_{i\in I^+} a_{ij}\bar{y}_i\bar{x}_j\\ &= \sum_{i\in I^+}\sum_{j\in J} a_{ij}\bar{y}_i\bar{x}_j = \sum_{i\in I^+}\bar{y}_i\sum_{j\in J} a_{ij}\bar{x}_j\\ &\geq \sum_{i\in I^+} b_i\bar{y}_i = \sum_{i\in I} b_i\bar{y}_i.\end{aligned}$$

The first inequality holds by (2b), non-negativity of $\bar{x}$ and the definition of $\sum_{j\in J}$. The first equality is by the definition of I^+ and the second equality holds by the definition of $\sum_{i\in I^+}$. The third equality holds because of non-negativity of a_{ij}, $\bar{x}$ and $\bar{y}$, and by the definitions of $\sum_{i\in I^+}$ and $\sum_{j\in J}$. The fourth equality holds by the definition of $\sum_{j\in J}$ and the second inequality holds by (1b). The last equality is by the definition of I^+. □