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On the core of traveling salesman games

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ABSTRACT

We define excess rate to study the core of traveling salesman games from a perspective of optimization, propose a new variant of the traveling salesman problem, and build a link between the two problems. An exact formula for the lowest achievable excess rate is found, which explains the existence of core emptiness. We then develop an implementable method to check whether empty core exists in general case. The results apply for both symmetric and asymmetric traveling salesman games.

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1. Introduction

A traveling salesman game (TSG) is motivated by the scenario that a set of customers receive their deliveries on the same route beginning and ending at the same supplier, and that these customers should share the delivery cost in a fair way such that no one has incentive to split from this shared route. Formally, a TSG is defined as follows. Given a complete graph $G = (V, E)$, where node set $V = \{0\} \cup N$ and E is the set of edges connecting every pair of nodes, 0 is the home node and N is the set of player nodes with $N = \{1, 2, \ldots, n\}$. Each arc $(i, j) \in E$ has cost $c_{ij} \geq 0$. We assume that the triangle inequality holds for the cost function, i.e. $c_{ik} \leq c_{ij} + c_{jk}$ for any $i, j, k \in V$. The cost function of a coalition of player nodes, $c(S)$, is the cost incurred by visiting all player nodes in S and returning to the home node, i.e. the optimal cost of the traveling salesman problem (TSP) on set $\{0\} \cup S$. An optimal route of a TSP on V is given with known cost $c(N)$. The cost of this route should be shared by all players and each player i has a cost allocated c_i. The core of this cooperative game is defined such that the costs allocated should satisfy

(i) $\sum_{i \in N} c_i = c(N)$;
(ii) $\sum_{i \in S} c_i \leq c(S)$ for all $S \subseteq N$.

Condition (i) is a budget balance guarantee that requires the total travel cost to be allocated completely. Condition (ii) is the stable allocation criterion stating that no subset of players could obtain lower cost by acting on its own.

Tamir [17] proved that a TSG with four or less players always has a non-empty core. This conclusion was extended to a TSG with five players by [10]. However, a TSG may have an empty core. Different example games of six players with an empty core were presented in [17,3]. A class of cost matrices were studied in [15,13] to define TSG with non-empty cores. But in a general case, testing the core non-emptiness of a given TSG is NP-hard [13].

Due to core emptiness, Faigle et al. [3] proposed an ϵ-approximation core by replacing condition (ii) by

(ii′) $\sum_{i \in S} c_i \leq (1 + \epsilon)c(S)$ for all $S \subseteq N$.

The parameter ϵ measures the closeness of a cost allocation to the core, and $\epsilon = 0$ means a stable allocation in the core. They developed a cost allocation method based on the duality of the Held–Karp relaxation of TSP. This work has been extended to the asymmetric TSG by [18]. Both of these papers resulted in stable allocation with a budget balance gap equal to the integrality gap of the corresponding TSP.

The TSP integrality gap and core emptiness seem to have some natural relationship, however, computational examples [18] showed that the core can be either empty or non-empty when the integrality gap exists. In this note, we investigate the property of the TSG core from a new perspective and build links to a new network optimization problem. This contributes in three aspects. First, an exact formula is obtained for the lowest ϵ achievable by an allocation, which reveals the reason of the core emptiness that has confused researchers for decades. Second, implementable methods naturally follow to check whether a general TSG has an empty core or not. Third, a promising direction for fair allocation of TSG is provided.

The remainder of this paper is organized as follows. In Section 2, we define the excess rate, which is closely related to the core

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emptiness. An explicit formula for the lowest achievable excess rate is derived. Based on findings in Section 2, Section 3 provides two ways to check core emptiness, either applying boundary properties for special networks or solving a TSP variant in general. We end with discussions on developing a general method for a fair allocation, and leave it as an open question.

2. Excess rate

Following the definition of an ϵ-approximation core [3], we define the *excess rate* as the maximum portion of over-allocated cost with respect to the stand alone cost among all the subsets $S \subseteq N$ for a given cost allocation, also denoted as ϵ, i.e. $\epsilon = \max_{S\subseteq N}\{\frac{\sum_{i\in S} c_i}{c(S)} - 1\}$. Denote ϵ^* as the lowest achievable excess rate among all possible allocations. If $\epsilon^* > 0$, the core of the TSG is empty; otherwise, a non-empty core exists. We then consider an optimization problem to find a cost allocation with the lowest possible excess rate. The problem is formulated below, where core conditions of TSG work as constraint sets, the excess rate is to be minimized and ϵ and c_i are decision variables.

(P) min ϵ

$$\text{s.t.}\quad c(S)\epsilon - \sum_{i\in S} c_i \geq -c(S) \quad \forall S\subseteq N \tag{1}$$

$$\sum_{i\in N} c_i = c(N) \tag{2}$$

c_i unrestricted $\quad \forall i \in N$.

Let w_S and v be dual variables of constraint sets (1) and (2). We then have the dual problem defined as

$$(D)\ \max\ c(N)v - \sum_{S\subseteq N} c(S)w_S$$

$$\text{s.t.}\quad v - \sum_{i\in S} w_S = 0 \quad \forall i\in N \tag{3}$$

$$\sum_{S\subseteq N} c(S)w_S = 1 \tag{4}$$

$w_S \geq 0 \quad \forall S\subseteq N$.

After substituting Eq. (4) in the objective, this dual problem is equivalent to (D') with the same constraints.

(D') max v.

At optimality, we have v^* and w_S^*. Because of constraint (3), $\sum_{i\in S} w_S^* = v^*\ \forall i$. Note that in the optimal solution to the original problem (P), $\epsilon^* = c(N)v^* - 1$.

Now define $k_S = \frac{w_S^*}{v^*}$, and then $\sum_{S:i\in S} k_S = 1\ \forall i$. Denote $|S|$ as the number of nodes in S, then w_S^* can be expressed as

$$w_S^* = \frac{\sum_{i\in S} k_S v^*}{|S|}. \tag{5}$$

Substitute Eq. (5) into constraint (4). After minor algebraic transformation we obtain

$$\frac{1}{v^*} = \sum_{S\subseteq N} c(S)\frac{\sum_{i\in S} k_S}{|S|} = \sum_{i\in N}\sum_{S:i\in S}\frac{c(S)}{|S|}k_S. \tag{6}$$

Since Problem (D') is to maximize v, it is equivalent to minimizing (6) with k_S unknown. Formally, the equivalent model is

$$(D'')\ \min \sum_{i\in N}\sum_{S:i\in S}\frac{c(S)}{|S|}k_S$$

$$\text{s.t.}\quad \sum_{S:i\in S} k_S = 1 \quad \forall i\in N \tag{7}$$

$k_S \geq 0 \quad \forall S\subseteq N$.

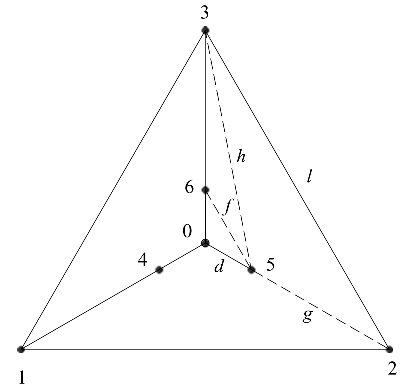

Fig. 1. A sample network with positive excess rate.

To understand the meaning of Problem (D''), we need to consider the following problem first.

Problem. Long-run Traveling Salesman Problem (LTSP)

Given: A set of player nodes N and a home node 0.

Objective: To find a set of tours and number of visits for each tour, such that the average cost per visit for all nodes in N is minimized.

Constraints: Each tour is a Hamilton path both starting from and ending at the home node. A single tour cannot stop at the same player node more than once. All player nodes have the same number of visits. Note, there is no limit on the total number of visits of a node.

An example is given to show the solution of a LTSP. The same network was presented by [3] to show that a TSG may have an empty core.

Example. Given a complete graph in Fig. 1, an equilateral triangle with three vertices, 1, 2 and 3, and a center 0, has edge lengths $l = \sqrt{3}$. Three points, 4, 5 and 6, are located at same distance, $d = \frac{1}{4}$, from the center. The following length information can be calculated: $g = \frac{3}{4}$, $f = \frac{\sqrt{3}}{4}$ and $h = \sqrt{1+d+d^2} = \frac{\sqrt{21}}{4}$.

0 is the home node. An optimal TSP route for this network is 0-4-1-5-2-3-6-0, which requires the total visit cost to be 5.63. However, the optimal solution to LTSP is that nodes are each visited twice through three tours respectively: 0-4-1-2-5-0, 0-5-2-3-6-0 and 0-6-3-1-4-0. In this solution, each node (except 0) is visited twice, and the average total visit cost is $0.5(3.73 + 3.73 + 3.73) = 5.60$. This example shows that using a TSP-optimal tour for the entire network to visit nodes every time may not be the optimal strategy in the long run.

In the optimal solutions of LTSP, a single tour with nodes consisting of subset S should be an optimal TSP tour of S, so it has cost $c(S)$. Suppose in a feasible solution of LTSP, each player node is visited T times and a tour of S is used t_S times. For a node i, denote its visit cost via the tour of S by $F(c(S), i)$. $F(c(S), i)$ can be obtained by any allocation method, and $\sum_{i\in S} F(c(S), i) = c(S)$. Thus, the average per visit cost to visit all nodes in N is

$$\frac{1}{T}\sum_{S\subseteq N} c(S)t_S = \sum_{S\subseteq N}\left(\sum_{i\in S} F(c(S), i)\right)\frac{t_S}{T}$$
$$= \sum_{i\in N}\sum_{S:i\in S} F(c(S), i)\frac{t_S}{T}.$$

Since the average cost to visit is independent of the allocation method, we could, for example, divide the cost evenly among all nodes in the subset S, i.e. $F(c(S), i) = \frac{c(S)}{|S|}$. Also, for each node i,

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