Operations Research Letters 43 (2015) 378–383

Contents lists available at ScienceDirect

Operations Research Letters

journal homepage: www.elsevier.com/locate/orl

Improved semidefinite approximation bounds for nonconvex nonhomogeneous quadratic optimization with ellipsoid constraints

Yong Hsia[a], Shu Wang[a], Zi Xu[b,*]

[a] State Key Laboratory of Software Development Environment, LMIB of the Ministry of Education, School of Mathematics and System Sciences, Beihang University, Beijing, 100191, PR China

[b] Department of Mathematics, Shanghai University, Shanghai 200444, PR China

ARTICLE INFO

Article history:
Received 30 September 2014
Received in revised form
6 May 2015
Accepted 6 May 2015
Available online 14 May 2015

Keywords:
Quadratic constrained quadratic programming
Semidefinite programming relaxation
Approximation algorithm

ABSTRACT

We study the problem of approximating nonconvex quadratic optimization with ellipsoid constraints (ECQP) and establish a new semidefinite approximation bound, which greatly improves Tseng's result (Tseng, 2003). As an application, we strictly improve the approximation ratio for the assignment-polytope constrained quadratic program. Finally, based on a randomized algorithm, we obtain a new approximation bound for (ECQP) which is sharp in the order of the number of the ellipsoid constraints.

1. Introduction

In this paper, we consider the following nonconvex quadratic optimization problem with ellipsoid constraints:

$$\begin{aligned} \min_{x\in\mathbb{R}^n}\ & f(x) = x^TAx + 2b^Tx \\ \text{s.t.}\ & \|F^kx+g^k\|^2 \le 1, \quad k=1,\ldots,m, \end{aligned} \tag{ECQP}$$

where $A \in \mathbb{R}^{n\times n}$ symmetric, $F^k \in \mathbb{R}^{r^k\times n}$, $b \in \mathbb{R}^n$, $g^k \in \mathbb{R}^{r^k}$, $r^k \ge 1$ and $\|\cdot\|$ denotes the Euclidean norm. Generally, this problem is NP-hard. To avoid trivial cases, we assume the Slater condition holds, i.e., the feasible region of (ECQP) has an interior point. With a proper transformation if necessary, we first make the following assumption.

Assumption 1.1. The origin 0 is in the interior of the feasible region of (ECQP), that is,

$$\|g^k\| < 1, \quad k=1,\ldots,m.$$

(ECQP) can be homogenized as

$$\min_{x\in\mathbb{R}^{n+1}} \sum_{i=1}^{n+1}\sum_{j=1}^{n+1} B_{ij}x_ix_j \tag{1}$$

$$\text{s.t.}\ \sum_{i=1}^{n+1}\sum_{j=1}^{n+1} B^k_{ij}x_ix_j \le 0, \quad k=1,\ldots,m, \tag{2}$$

$$x_{n+1} = 1, \tag{3}$$

where

$$B = \begin{bmatrix} A & b \\ b^T & 0 \end{bmatrix},$$

$$B^k = \begin{bmatrix} (F^k)^TF^k & (F^k)^Tg^k \\ (g^k)^TF^k & \|g^k\|^2-1 \end{bmatrix}, \quad k=1,\ldots,m.$$

By letting $X = xx^T$ and dropping the rank one constraint, the semidefinite programming relaxation of (ECQP) can be written as follows.

$$\begin{aligned} \min\ & B\bullet X \\ \text{s.t.}\ & B^k\bullet X \le 0,\ k=1,\ldots,m, \\ & X_{n+1,n+1} = 1, \quad X \succeq 0,\ X\in\mathbb{R}^{(n+1)\times(n+1)}. \end{aligned} \tag{SDP}$$

In addition, we need to make the following assumption for (SDP) throughout this paper.

* Corresponding author.
E-mail addresses: dearyxia@gmail.com (Y. Hsia), wangshu.0130@163.com (S. Wang), xuzi@shu.edu.cn (Z. Xu).

http://dx.doi.org/10.1016/j.orl.2015.05.002
0167-6377/

Assumption 1.2. (SDP) has an optimal solution X^*.

Let $v(\cdot)$ denote the optimal value of problem $(\cdot)$. Obviously, we have

$v(\text{SDP}) \le v(\text{ECQP}) \le f(0) = 0,$

where the equality of the first inequality holds if and only if rank $(X^*) = 1$ with X^* being an optimal solution of (SDP), and the second inequality follows from Assumption 1.1. Throughout this paper, we call τ the approximation ratio for the minimization problem (ECQP) if

$v(\text{ECQP}) \le \tau \cdot v(\text{SDP}).$

The following theorem establishes an approximation bound for (ECQP).

Theorem 1.3 ([11]). *Under Assumptions 1.1 and 1.2, a feasible solution x for (ECQP) can be generated in polynomial time satisfying*

$$f(x) \le \frac{(1-\gamma)^2}{(\sqrt{m}+\gamma)^2} \cdot v(\text{SDP}), \tag{4}$$

where $\gamma := \max_{k=1,\ldots,m} \|g^k\|$.

When $b = 0$ and $g^k = 0$ for $k = 1, \ldots, m$ and $\sum_{k=1}^m (F^k)^T F^k$ is positive definite, it was shown in [6] that a feasible solution x can be generated from (SDP) satisfying

$$f(x) \le \frac{1}{2\ln(2(m+1)\mu)} \cdot v(\text{SDP}) \tag{5}$$

with $\mu := \min\{m+1, \max_{k=1,\ldots,m} \operatorname{rank}((F^k)^T F^k)\}$. In particular, when (ECQP) has a ball constraint, $\mu = \min\{m+1, n\}$. Also for this special case, Ye and Zhang (Corollary 2.6 in [13]) showed that a feasible solution x satisfying

$$f(x) \le \frac{1}{\min\{m-1, n\}} \cdot v(\text{SDP}),$$

can be found in polynomial time. For more detailed results related to this special case, we refer to the survey paper [5]. When $A \preceq 0$, $b = 0$ but allowing nonzero $\|g^k\|$ for $k = 1, \ldots, m$, Ye showed in [12] that a feasible solution $\widetilde{x}$ can be randomly generated such that

$$E(\widetilde{x}^T A \widetilde{x}) \le \frac{(1-\max\limits_k \|g^k\|)^2}{4\ln(4mn \cdot \max\limits_k(\operatorname{rank}((F^k)^T F^k)))} \cdot v(\text{SDP}). \tag{6}$$

To be mentioned, the n in the denominator should be $n+1$ according to Ye's proof in [12].

By directly applying the rank reduction result of Shapiro–Barvinok–Pataki (see Theorem 2.1), we can see that the denominators of (5) and (6) can be strengthened to $2\ln(2(m+1) \cdot \min\{\sqrt{2m}+1, \max_k \operatorname{rank}((F^k)^T F^k)\})$ and $4\ln(4m \cdot \min\{\sqrt{2m}, n+1\} \cdot \min\{\sqrt{2m}, \max_k \operatorname{rank}((F^k)^T F^k)\})$, respectively, see Theorem 1.1 in [9] and the following remarks. However, the same approach cannot be trivially applied to improve the approximation ratio for (ECQP).

In Section 2 of this paper, based on a new analysis, we establish a sharper semidefinite approximation bound for (ECQP). More precisely, from an optimal solution of (SDP), a feasible solution x for (ECQP) can be generated, which satisfies that

$$f(x) \le \frac{(1-\gamma)^2}{\left(\sqrt{\widetilde{r}}+\gamma\right)^2} \cdot v(\text{SDP}),$$

where $\widetilde{r} = \min\left\{\left\lceil \frac{\sqrt{8m+17}-3}{2} \right\rceil, n+1\right\}$ and γ is defined the same as in Theorem 1.3. This bound improves the result shown in Theorem 1.3 in the order of m, i.e., from $O(1/m)$ to $O(1/\sqrt{m})$.

As an application of (ECQP), in Section 3, we consider the assignment-polytope constrained QP problem (AQP) and show a strictly improved approximation bound compared to Fu et al.'s result [3]. Although, it is claimed in [12] that this ratio can be improved from $1/O(n^3)$ to $1/O(n^2\log(4n^4))$, the analysis technique therein only works for a very special case of (AQP).

In Section 4, by a similar randomized algorithm proposed in Nemirovski et al.'s paper [6], we give a further improved approximation bound for (ECQP). A feasible solution x for (ECQP) can be generated, which satisfies that

$$f(x) \le \frac{(1-\gamma)^2}{\left(\sqrt{M}+\gamma\right)^2} \cdot v(\text{SDP}),$$

where $M = 2\ln(100m \cdot \min\{\left\lceil \frac{\sqrt{8m+17}-3}{2} \right\rceil, \max_{k=1,\ldots,m}(\operatorname{rank}(A^k))\})$ and γ is defined the same as in Theorem 1.3. This bound improves the result shown in Theorem 1.3 in the order of m, i.e., from $O(1/m)$ to $O(1/\ln m)$. Moreover, the new bound is sharp in the order of m in general.

Notations. Throughout the paper, $A \succeq 0$ stands for the matrix A is positive semidefinite, $A \bullet B = \sum_{i,j=1}^n a_{ij}b_{ij}$ is the inner product of two matrices A, B. $\operatorname{Tr}(X)$ denotes the trace of the matrix X. Let $\mathbb{R}^n$ and S_+^n be the n-dimensional vector space and $n \times n$ positive semidefinite symmetric matrix space, respectively. The notation ":=" denotes "define".

2. Improved approximation bound

In this section, we establish a sharper approximation bound for (ECQP). Before presenting the main result, we first restate the well-known Shapiro–Barvinok–Pataki rank reduction result for (SDP) due to Shapiro [8], Barvinok [1] and Pataki [7].

Theorem 2.1 ([8,1,7]). *Let r be a positive integer. Suppose that (SDP) is solvable and*

$$m+1 \le (r+2)(r+1)/2 - 1. \tag{7}$$

Then (SDP) has a solution X^* for which $\operatorname{rank}(X^*) \le r$.

It can be easily verified that (7) is equivalent to

$$r \ge \left\lceil \frac{\sqrt{8m+17}-3}{2} \right\rceil := r_0. \tag{8}$$

Moreover, an algorithm called "algorithm RED" is proposed in [2] to find such a solution with rank less than or equal to r_0. Next we introduce the following rank-1 decomposition theorem proposed by Sturm and Zhang in [10].

Theorem 2.2 ([10]). *Let X be a positive semidefinite matrix of rank r. Then, $B \bullet X \le 0$ if and only if there is a rank-one decomposition*

$$X = \sum_{i=1}^r w_i w_i^T$$

such that $w_i^T B w_i \le 0$ for $i = 1, \ldots, r$.

Let X^* be an optimal solution of (SDP) and r be the rank of X^*. According to Theorem 2.1, we can assume r satisfies (8).