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Pricing volatility derivatives under the modified constant elasticity of variance model

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ABSTRACT

This paper studies volatility derivatives such as variance and volatility swaps, options on variance in the modified constant elasticity of variance model using the benchmark approach. The analytical expressions of pricing formulas for variance swaps are presented. In addition, the numerical solutions for variance swaps, volatility swaps and options on variance are demonstrated.

1. Introduction

This paper considers the modified constant elasticity of variance (MCEV) model, which is an extension to the Black–Scholes–Merton model and the stylized minimal market model; see [19]. The standard CEV model was originally introduced by [9]. The main advantages of using the CEV model are that it can account for the implied volatility smile and smirk by capturing the leverage effect.

The pricing of different kinds of options under the constant elasticity of variance (CEV) model has provided interesting and challenging research topics; see e.g. [2,11,17,12,19]. The latter paper modeled the growth optimal portfolio (GOP) under the real world probability measure, where it is referred to as the MCEV model. The current paper will study volatility derivatives under this model.

Since the S&P 500 volatility index VIX was introduced in 1993, there have been more and more volatility derivatives tradable on the exchanges or over the counter. The VIX index can be theoretically interpreted as the standardized risk-neutral expected realized variance; see [4,7]. Recent literature discussing volatility derivatives include [13,5,6,16,8].

We will apply the benchmark approach, documented in [19], which uses the GOP as the numéraire so that the contingent claims will be priced under the real world probability measure. This avoids the restrictive assumption on the existence of an equivalent risk neutral probability measure. As argued in [19], this measure seems not to exist for realistic models and does not exist for the MCEV model. In the following, we derive closed-form formulas for variance swaps under the MCEV model and show numerical results for volatility derivatives.

2. Volatility derivatives

A variance swap is a forward contract on annualized variance. Let $\sigma^2_{0,T}$ denote the realized annualized variance of the log-returns of a diversified equity index or related futures over the life of the contract such that

$$\sigma^2_{0,T} := \frac{1}{T}\int_0^T \sigma^2_u du. \tag{2.1}$$

Assume that one can trade the underlying futures or index price at discrete times $t_i = i\Delta$ for $i \in \{0, 1, \ldots\}$ with time step size $\Delta > 0$. The period Δ between two successive potential trading times is typically the length of one day. $S^{\delta_*}_{t_i}$ denotes the index price at time t_i for $i \in \{0, 1, 2, \ldots\}$.

Let $(\Omega, \mathcal{A}_T, \underline{\mathcal{A}}, \mathcal{P})$ denote the underlying filtered probability space satisfying usual conditions. Here $\mathcal{P}$ is the real world probability measure and $\underline{\mathcal{A}} = (\mathcal{A}_t)_{t\in[0,T]}$ the respective filtration. For simplicity, assume throughout the paper that the interest rate $r > 0$ is constant. Furthermore, we assume that the index is the GOP $S^{\delta_*}_t$, also called benchmark of the market. We call any price or payoff

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denominated in units of the GOP the respective benchmarked price. We employ in this paper the real world pricing formula, which provides for a replicable $\mathcal{A}_{\bar{T}}$-measurable contingent claim $H_{\bar{T}}$ with $E(\frac{|H_{\bar{T}}|}{S^{\delta_*}_{\bar{T}}}) < \infty$ the real world pricing formula

$$V_t = S^{\delta_*}_t E\left[\frac{H_{\bar{T}}}{S^{\delta_*}_{\bar{T}}}\middle|\mathcal{A}_t\right] \tag{2.2}$$

for all $t \in [0, \bar{T}]$, $\bar{T} \in [0, T]$; see [19].

Let K_v denote the delivery price for realized variance and L the notional amount of the swap in dollars per annualized variance point. Then, the payoff of the variance swap at expiration time T is given by $L(\sigma^2_{0,T} - K_v)$.

A volatility swap is a forward contract on annualized volatility. Let K_s denote the annualized volatility delivery price and L the notional amount of the swap in dollar per annualized volatility point. Then, the payoff function of the volatility swap is given by $L(\sigma_{0,T} - K_s)$, where $\sigma_{0,T} = \sqrt{\sigma^2_{0,T}}$.

Additionally, we will consider the payoffs of call options on variance, defined by $(\sigma^2_{0,T} - K)^+$, as well as, the payoffs of put options on variance, defined by $(K-\sigma^2_{0,T})^+$, where $a^+ = \max(0, a)$.

3. Modified constant elasticity of variance model

As shown in [14], the MCEV model for the GOP is obtained when the volatility of the GOP takes the form

$$|\theta_t| = (S^{\delta_*}_t)^{a-1}\psi, \tag{3.1}$$

for $t \in [0, \infty)$ with exponent $a \in (-\infty, \infty)$, $a \neq 1$, and scaling parameter $\psi > 0$. From [14], recall that the discounted GOP satisfies the SDE

$$dS^{\delta_*}_t = \left(rS^{\delta_*}_t + (S^{\delta_*}_t)^{2a-1}\psi^2\right)dt + (S^{\delta_*}_t)^a\psi dW(t), \tag{3.2}$$

for $t \in [0, T]$. Now set $X_t = (S^{\delta_*}_t)^{2(1-a)}$. Then we have

$$dX_t = k(\vartheta - X_t)dt + \sigma\sqrt{X_t}dW(t), \tag{3.3}$$

where $k = -2(1-a)r$, $\vartheta = -\frac{\psi^2(3-2a)}{2r}$, $\sigma = 2\psi(1-a)$. Note that X_t is a space–time changed squared Bessel process of dimension $\delta = \frac{3-2a}{1-a}$; see [15].

4. Explicit formula for variance swaps

Due to (2.2) the value of a variance swap $V_v(t, S^{\delta_*}_t)$ at time $t = 0$ is given by:

$$V_v(0, S^{\delta_*}_0) = S^{\delta_*}_0 E\left[\frac{L(\sigma^2_{0,T} - K_v)}{S^{\delta_*}_T}\right] = S^{\delta_*}_0 LE\left[\frac{\sigma^2_{0,T}}{S^{\delta_*}_T}\right] - S^{\delta_*}_0 LK_v E\left[\frac{1}{S^{\delta_*}_T}\right]. \tag{4.1}$$

Hence, the evaluation of the price of a variance swap can be reduced to the problem of calculating the expected value $E[\frac{\sigma^2_{0,T}}{S^{\delta_*}_T}]$ of the benchmarked realized annualized variance and the zero coupon bond $B_T(0, S^{\delta_*}_0) = S^{\delta_*}_0 E[\frac{1}{S^{\delta_*}_T}]$.

As followed from [18], the price of a zero-coupon bond $B_T(t, S^{\delta_*}_t)$, calculated at time t with maturity T under the given MCEV model, equals

$$B_T(t, S^{\delta_*}_t) = e^{-r(T-t)}\chi^2\left(\Upsilon_T; \frac{1}{1-a}\right), \tag{4.2}$$

where

$$\Upsilon_T = \frac{2r}{|\theta_t|^2(1-a)[1-\exp\{-2(1-a)r(T-t)\}]} \tag{4.3}$$

for $t \in [0, T]$ and $\chi^2(u, \nu) = 1 - \frac{\Gamma(\frac{u}{2}; \frac{\nu}{2})}{\Gamma(\frac{\nu}{2})}$ for $u \geq 0$ and where $\Gamma(\alpha)$ for $\alpha > -1$ is the gamma function, and $\Gamma(., .)$ is the incomplete gamma function; see [19].

Furthermore, we have

$$E\left[\frac{\sigma^2_{0,T}}{S^{\delta_*}_T}\right] = \frac{\psi^2}{T}E\left[\frac{\int_0^T (S^{\delta_*}_s)^{2(a-1)}ds}{S^{\delta_*}_T}\right] = \frac{\psi^2}{T}E\left[\frac{\int_0^T \frac{1}{X_s}ds}{X_T^{\frac{1}{2(1-a)}}}\right]. \tag{4.4}$$

Similar to Proposition 8.1 in [8], we can prove:

Lemma 4.1. *Let $X = \{X_t : t \in [0, T]\}$ satisfy the SDE (3.3) and set $\beta = 1 + m - \frac{1}{2(1-a)} + \nu/2$, $m = \frac{1}{2}(\frac{2k\vartheta}{\sigma^2} - 1)$, $\nu = \frac{2}{\sigma^2}\sqrt{(k\vartheta - \frac{\sigma^2}{2})^2 + 2\mu\sigma^2}$, $\mu > 0$ and $X_0 = x > 0$. Then if $m > \frac{1}{2(1-a)} - \frac{\nu}{2} - 1$, we have*

$$E\left[\frac{\int_0^T \frac{ds}{X_s}}{X_T^{\frac{1}{2(1-a)}}}\right] = -\frac{d}{d\mu}\frac{1}{2^\nu x^m}e^{-\frac{2kx}{\sigma^2(e^{kT}-1)}+kmt}\left(\frac{2ke^{kT}}{(e^{kT}-1)\sigma^2}\right)^{-m+\frac{1}{2(1-a)}-\frac{\nu}{2}} \times \left(\frac{4k^2x}{\sigma^4\sinh^2\left(\frac{kT}{2}\right)}\right)^{\nu/2}\frac{\Gamma\left(1+m-\frac{1}{2(1-a)}+\frac{\nu}{2}\right)}{\Gamma(1+\nu)} \times {}_1F_1\left(\beta, 1+\nu, \frac{2kx}{\sigma^2(e^{kT}-1)}\right)\Bigg|_{\mu=0}. \tag{4.5}$$

Here the function ${}_1F_1(., ., .)$ is the confluent hypergeometric function, see [8]. In the working paper version of the current paper the interested reader can find an alternative proof to [8].

Now, we give an example for variance swaps. The values for the parameters of the model are set to $k = 0.052$, $a = \frac{2}{3}$, $\vartheta = 24.0385$, $\psi = 1.5$, $m = \frac{3}{4}$, $\sigma = 0.3162$, $x = 1$, $L = 1$ million dollars and $K_v = 1$.

Table 4.1 displays the prices of variance swaps for various maturities.

Table 4.1
Prices of variance swaps.

Maturities	Prices of variance swaps
1/6	1.58587
0.25	1.85638
0.5	2.51737
1	1.83401
1.5	0.89961
2	0.301646

5. Options on variance

According to (2.2), the value of a call option on variance at time zero is given by:

$$C_v(0, S^{\delta_*}_0) = S^{\delta_*}_0 E\left[\frac{(\sigma^2_{0,T} - K)^+}{S^{\delta_*}_T}\right] = S^{\delta_*}_0 E\left[\left(\frac{\sigma^2_{0,T}}{S^{\delta_*}_T} - \frac{K}{S^{\delta_*}_T}\right)^+\right]. \tag{5.1}$$