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Constructing general dual-feasible functions

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ABSTRACT

Dual-feasible functions have proved to be very effective for generating fast lower bounds and valid inequalities for integer linear programs with knapsack constraints. However, a significant limitation is that they are defined only for positive arguments. Extending the concept of dual-feasible function to the general domain and range $\mathbb{R}$ is not straightforward. In this paper, we propose the first construction principles to obtain general functions with domain and range $\mathbb{R}$, and we show that they lead to non-dominated maximal functions.

1. Introduction

In [9], the authors showed that extending the concept of dual-feasible function to domain and range $\mathbb{R}$ is far from straightforward. Many properties of standard dual-feasible functions are lost in this process, thus complicating the task of deriving strong non-dominated functions. In this paper, we introduce the first general construction principles that allow to generate dual-feasible functions with domain and range $\mathbb{R}$. These principles can be seen as general methods defining families of different dual-feasible functions. We show in particular that these principles lead to families of non-dominated dual-feasible functions from which strong lower bounds and inequalities can be obtained for integer linear optimization problems with knapsack constraints and general coefficients. To complete the analysis of these general construction principles, we describe and analyse specific instances of functions obtained from each principle.

In [4], Gomory and Johnson showed how valid inequalities can be obtained from knapsack constraints by analysing corner polyhedra, and how to use them to generate other valid inequalities by interpolation, particularly in their Theorem 3.1. This idea was further explored by Dash et al. in [3]. However, both the approaches by Gomory and Johnson [4] and Dash et al. [3] require rational problem data, while the contributions described in our paper do not impose such prerequisites. Furthermore, although Theorem 1.5 of [4], referenced by the proof of Theorem 3.1 in the same paper, provides a characterization of subadditive functions on a subgroup of the unit interval with addition modulo 1, this theorem does not explain how to construct these functions, like it is done in our contribution for superadditive functions.

In Section 2, we recall some of the definitions related to standard and general dual-feasible functions. The construction principles are introduced and analysed in Section 3. To further illustrate these general construction principles, we introduce in Section 4 specific examples of dual-feasible functions with domain and range $\mathbb{R}$ obtained by applying each procedure.

2. General dual-feasible functions

The vast majority of the dual-feasible functions described in the literature is defined for positive arguments only. Most of the time, these functions are declared on the domain $[0, 1]$, although their extension to the domain $[0, C]$ with a constant $C > 0$ is usually straightforward. The formal definition of these standard dual-feasible functions stands as follows.

Definition 1. A function $f : [0, 1] \to [0, 1]$ is a *dual-feasible function (DFF)*, if for any finite set $\{x_i \in \mathbb{R}_+ : i \in I\}$ of nonnegative numbers, it holds that

$$\sum_{i\in I} x_i \le 1 \Longrightarrow \sum_{i\in I} f(x_i) \le 1.$$

We will use the term *general dual-feasible function* to refer to a dual-feasible function whose domain is not restricted to positive arguments, but that considers instead the domain $\mathbb{R}$ of any real value. In [9], we showed that this generalization is not straight-

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forward. Indeed, several properties that apply to standard functions are lost when one goes into the domain of real arguments. The counterpart is that general dual-feasible functions can be used on problems whose knapsack constraints have general coefficients and hence their applicability increases significantly. General dual-feasible functions are defined as follows.

Definition 2. A function $f : \mathbb{R} \to \mathbb{R}$ is a *general dual-feasible function*, if for any finite set $\{x_i \in \mathbb{R} : i \in I\}$ of real numbers, it holds that

$$\sum_{i\in I} x_i \le 1 \Longrightarrow \sum_{i\in I} f(x_i) \le 1. \tag{1}$$

Maximality is an important property that distinguishes dominated from non-dominated functions. In practice, a (general) dual-feasible function f is *maximal*, if there is no other (general) dual-feasible function g with $f(x) \le g(x)$ for all possible values of x. We recall in the sequel the properties of standard maximal dual-feasible functions from $[0, 1] \to [0, 1]$. For a function $f : [0, 1] \to [0, 1]$ to be a maximal dual-feasible function (MDFF), it is necessary and sufficient [5] that f is symmetric:

$$f(x) + f(1-x) = 1 \quad \text{for all } x \in [0, 1/2], \tag{2}$$

$f(0) = 0$, and f satisfies the superadditivity condition:

$$f(x_1 + x_2) \ge f(x_1) + f(x_2)$$
$$\text{for all } x_1, x_2 \text{ with } 0 < x_1 \le x_2 < 1/2 \text{ and } x_1 + x_2 \le 2/3.$$

For general dual-feasible functions, the symmetry rule (2) becomes

$$f(x) + f(1-x) = 1, \quad \text{for all } x \le 1/2. \tag{3}$$

An example of how standard dual-feasible functions may be different from general dual-feasible functions is the fact that symmetry (3), which is an important property of standard MDFFs, has not to be necessarily fulfilled by a general MDFF. In [8], we stated the conditions for a general dual-feasible function to be maximal. We recall these conditions next.

Theorem 1 (*[8]*)**.** *Let $f : \mathbb{R} \to \mathbb{R}$ be a given function.*

(a) *If f satisfies the following conditions, then f is a general MDFF:*
   1. *$f(0) = 0$;*
   2. *f is superadditive, i.e., for all $x, y \in \mathbb{R}$, it holds that* $$f(x+y) \ge f(x) + f(y); \tag{4}$$
   3. *there is an $\varepsilon > 0$, such that $f(x) \ge 0$ for all $x \in (0, \varepsilon)$;*
   4. *f obeys the symmetry rule (3).*

(b) *If f is a general MDFF, then the above properties* (1)–(3) *hold for f, but not necessarily* (4).

(c) *If f satisfies the above conditions* (1)–(3)*, then f is monotone increasing.*

(d) *If the symmetry rule (3) holds and f obeys the inequality (4) for all $x, y \in \mathbb{R}$ with $x \le y \le \frac{1-x}{2}$, then f is superadditive.*

Proof. The proof provided in [8] is repeated here for the sake of clarity. The proof is made in the following order: First, we prove (c), then (b), (a), and finally (d) is proved.

(c) If f satisfies the first three conditions, then for any $x > 0$ setting $n := \lfloor x/\varepsilon \rfloor + 1$ yields $n \in \mathbb{N} \setminus \{0\}$ and $0 < x/n < \varepsilon$. Hence, we have $f(x/n) \ge 0$ and $f(x) \ge n \times f(x/n) \ge 0$. Therefore, the monotonicity follows immediately from $f(x_2) \ge f(x_1) + f(x_2 - x_1)$ for any $x_1, x_2 \in \mathbb{R}$ with $x_1 \le x_2$.

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be a general MDFF. We prove the properties 1. to 3. of part (a). One has $f(0) \le 0$ due to the defining condition (1) for general dual-feasible functions. On the other hand, $f(x) < 0$ for a certain $x \ge 0$ is impossible, because f is maximal and setting $f(x)$ to zero cannot violate the condition (1) for general dual-feasible functions. Assume that $f(x_1 + x_2) < f(x_1) + f(x_2)$ for certain $x_1, x_2 \in \mathbb{R}$. Define a function $g : \mathbb{R} \to \mathbb{R}$ as

$$g(x) := \begin{cases} f(x) & \text{if } x \neq x_1 + x_2 \\ f(x_1) + f(x_2) & \text{otherwise.} \end{cases}$$

Since f is a general MDFF, g must violate the defining condition for a general dual-feasible function. Suppose, one summand on the left part of (1) equals $x_1 + x_2$. Replacing it by two summands x_1 and x_2 leads to a violation if $x_1, x_2 \neq 0$, because of the definition of g. That is a contradiction.

It remains to give a counter example to the symmetry (3). For any constant $c \in [0, 1]$, the linear function $f : \mathbb{R} \to \mathbb{R}$ with $f(x) := cx$ is a general MDFF, but not symmetric for $c < 1$. It is a general dual-feasible function according to Definition 2, because for any $n \in \mathbb{N} \setminus \{0\}$ and numbers $x_1, \dots, x_n \in \mathbb{R}$ with $\sum_{i=1}^n x_i \le 1$, it holds that $\sum_{i=1}^n f(x_i) = c * \sum_{i=1}^n x_i \le c$. Suppose, there is a general dual-feasible function $g : \mathbb{R} \to \mathbb{R}$ with $g(x) \ge cx$ for all $x \in \mathbb{R}$ and $g(y) > cy$ for a certain $y \in \mathbb{R}$. Definition 2 implies $g(y) + g(-y) \le 0$. Since $g(-y) \ge f(-y)$, the contradiction $0 \ge g(y) + g(-y) > cy - cy = 0$ follows. Since f is not dominated by another general dual-feasible function, it is maximal. If $c < 1$ then $f(x) + f(1-x) = c < 1$, violating (3).

(a) The converse direction is to prove that if f satisfies conditions 1. to 4. of part (a), then f is a general MDFF. For any $n \in \mathbb{N}$ and $x_1, \dots, x_n \in \mathbb{R}$ with $\sum_{i=1}^n x_i \le 1$, the superadditivity condition 2. in part (a) yields $\sum_{i=1}^n f(x_i) \le f(\sum_{i=1}^n x_i)$. Let $x_0 := 1 - \sum_{i=1}^n x_i \ge 0$. Therefore, we have $f(x_0) \ge 0$ due to the already proved part (c). Because of $f(1 - x_0) + f(x_0) = 1$, it follows that f is a general dual-feasible function. Let $g : \mathbb{R} \to \mathbb{R}$ be a general dual-feasible function with $g(x) > f(x)$ for a certain $x \in \mathbb{R}$. Since g is a general dual-feasible function, one has $g(1-x) + g(x) \le 1$. It follows that $g(1-x) \le 1 - g(x) < 1 - f(x) = f(1-x)$ due to (3), hence g does not dominate f. Therefore, f is a general MDFF.

(d) If $x > 1/2$, then $z := 1 - x < 1/2$, and hence $f(z) + f(1-z) = 1$ due to (3). That implies $f(x) + f(1-x) = 1$. This symmetry will be assumed for the entire remaining proof. The condition $x \le y \le \frac{1-x}{2}$ implies $x + y \le 2/3$ and $x \le 1/3$, because $x \le \frac{1-x}{2}$ leads to $3x \le 1$ and therefore $x + y \le \frac{1+x}{2} \le \frac{1+1/3}{2} = \frac{2}{3}$. Obviously, the inequality (4) is valid if and only if it is true after exchanging x against y. Therefore, $x \le y$ can be enforced without loss of generality. Now we prove that the inequality (4) holds for all $x, y \in \mathbb{R}$, if it is true for all $x, y \in \mathbb{R}$ with $x + y \le 2/3$. If $x + y > 2/3$, then $y > 1/3$ due to $x \le y$. Hence, $1 - y < 2/3$ and $f(x) + f(1-y-x) \le f(1-y)$ according to the inequality (4). The symmetry (3) yields $f(x) + 1 - f(x+y) \le 1 - f(y)$, and hence $f(x) + f(y) \le f(x+y)$, as needed. Therefore, $x + y \le 2/3$ can be assumed in the rest of the proof, and hence $x \le \frac{1}{3} \le \frac{1-x}{2}$. If $y > \frac{1-x}{2}$, then let $z := 1 - x - y < \frac{1-x}{2}$. Due to the previous parts of the proof of point (d) and the prerequisites, the superadditivity rule (4) can be used, implying $f(x) + f(z) \le f(x+z)$. The symmetry rule (3) yields $f(x) + 1 - f(1-z) \le 1 - f(1-x-z)$, and hence $f(x) + f(1-x-z) = f(x) + f(y) \le f(1-z) = f(x+y)$. □

It is well known that standard dual-feasible functions generate solutions that are feasible for the dual of instances of the 1-dimensional cutting stock problem. The same happens with general dual-feasible functions for the case where the sizes of the items and the variables of the dual problem are not restricted in sign. Negative sizes may happen when it is possible to use a certain fixed quantity of extra space in containers in limited number (which can be seen as items of negative size).

Negative sizes also occur when balance constraints are considered. For example, let us consider a process assignment problem, where p processes have to be assigned to a minimum number of identical machines. Each process i has a given demand c_i in CPU,