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Disjunctive cuts for cross-sections of the second-order cone

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ABSTRACT

In this paper we study general two-term disjunctions on affine cross-sections of the second-order cone. Under some mild assumptions, we derive a closed-form expression for a convex inequality that is valid for such a disjunctive set, and we show that this inequality is sufficient to characterize the closed convex hull of all two-term disjunctions on ellipsoids and paraboloids and a wide class of two-term disjunctions – including split disjunctions – on hyperboloids. Our approach relies on the work of Kılınç-Karzan and Yıldız which considers general two-term disjunctions on the second-order cone.

1. Introduction

In this paper we consider the mixed-integer second-order conic set

$S := \{x \in \mathbb{L}^n : Ax = b, \ x_j \in \mathbb{Z} \ \forall j \in J\}$

where $\mathbb{L}^n$ is the n-dimensional second-order cone $\mathbb{L}^n := \{x \in \mathbb{R}^n : \|(x_1; \ldots; x_{n-1})\| \le x_n\}$, A is an $m \times n$ real matrix of full row rank, d and b are real vectors of appropriate dimensions, $J \subseteq \{1, \ldots, n\}$, and $\|.\|$ denotes the Euclidean norm. The set S appears as the feasible solution set or a relaxation thereof in mixed-integer second order cone programming problems. Because the structure of S can be very complicated, a first approach to solving

$$\sup \left\{d^\top x : \ x \in S\right\} \tag{1}$$

entails solving the relaxed problem obtained after dropping the integrality requirements on the variables:

$$\sup \left\{d^\top x : \ x \in C\right\} \quad \text{where } C := \left\{x \in \mathbb{L}^n : \ Ax = b\right\}.$$

The set C is called the natural *continuous relaxation* of S. Unfortunately, the continuous relaxation C is often a poor approximation to the mixed-integer conic set S, and tighter formulations are needed for the development of practical strategies for solving (1). An effective way to improve the approximation quality of the continuous relaxation C is to strengthen it with additional convex inequalities that are valid for S but not for the whole of C. Such valid inequalities can be derived by exploiting the integrality of the variables $x_j, \ j \in J$, and enhancing C with linear *two-term disjunctions* $l_1^\top x \ge l_{1,0} \ \vee \ l_2^\top x \ge l_{2,0}$ that are satisfied by all solutions in S. Valid inequalities that are obtained from disjunctions using this approach are known as *disjunctive cuts*. In this paper we study two-term disjunctions on the set C and give closed-form expressions for the strongest disjunctive cuts that can be obtained from such disjunctions.

Disjunctive cuts were introduced by Balas in the context of mixed-integer linear programming [3] and have since been the cornerstone of theoretical and practical achievements in integer programming. There has been a lot of recent interest in extending disjunctive cutting-plane theory from the domain of mixed-integer linear programming to that of mixed-integer conic programming [18,9,11,12,7,2]. Kılınç-Karzan [13] studied minimal valid linear inequalities for general disjunctive conic sets and showed that these are sufficient to describe the associated closed convex hull under a mild technical assumption. Bienstock and Michalka [6] studied the characterization and separation of linear inequalities that are valid for the epigraph of a convex, differentiable function whose domain is restricted to the complement of a convex set. On the other hand, several papers in the last few years have focused on deriving closed-form expressions for nonlinear convex inequalities that fully describe the convex hull of a disjunctive second-order conic set in the space of the original variables. Dadush et al. [10] and Andersen and Jensen [1] derived split cuts for ellipsoids and the second-order cone, respectively. Modaresi et al. extended these results to split disjunctions on cross-sections of the second-order cone [15] and compared the effectiveness of split cuts against conic MIR inequalities and extended formulations [16]. For disjoint two-term disjunctions on cross-sections of the second-order cone

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and under the assumption that $\{x \in C : l_1^\top x = l_{1,0}\}$ and $\{x \in C : l_2^\top x = l_{2,0}\}$ are bounded, Belotti et al. [5,4] proved that there exists a unique cone which describes the convex hull of the disjunction. They also identified a procedure for identifying this cone when C is an ellipsoid. Using the structure of minimal valid linear inequalities, Kılınç-Karzan and Yıldız [14] derived a family of convex inequalities which describes the convex hull of a general two-term disjunction on the whole second-order cone. In this paper, we pursue a similar goal: we study general two-term disjunctions on a cross-section C of the second-order cone, namely $C = \{x \in \mathbb{L}^n : Ax = b\}$. Given a disjunction $l_1^\top x \ge l_{1,0} \vee l_2^\top x \ge l_{2,0}$ on C, we let

$$C_1 := \left\{x \in C : l_1^\top x \ge l_{1,0}\right\} \quad \text{and} \quad C_2 := \left\{x \in C : l_2^\top x \ge l_{2,0}\right\}.$$

In order to derive the tightest disjunctive cuts that can be obtained for S from the disjunction $C_1 \cup C_2$, we study the closed convex hull $\overline{\text{conv}}(C_1 \cup C_2)$. In particular, we are interested in convex inequalities that may be added to the description of C to obtain a characterization of $\overline{\text{conv}}(C_1 \cup C_2)$. Our starting point is the paper [14] about two-term disjunctions on the second-order cone $\mathbb{L}^n$. We extend the main result of [14] to cross-sections of the second-order cone. Such cross-sections include ellipsoids, paraboloids, and hyperboloids as special cases. Our results generalize the work of [10,15] on split disjunctions on cross-sections of the second-order cone and [4] on disjoint two-term disjunctions on ellipsoids. We note here that general results on convexifying the intersection of a cross-section of the second-order cone with a non-convex cone defined by a single homogeneous quadratic were recently obtained independently in [8].

We first show in Section 2 that the continuous relaxation C can be assumed to be the intersection of a lower-dimensional second-order cone with a single hyperplane. In Section 3, we give a complete description of the convex hull of a homogeneous two-term disjunction on the whole second-order cone. In Section 4, we prove our main result, Theorem 3, characterizing $\overline{\text{conv}}(C_1 \cup C_2)$ under certain conditions. We end the paper with two examples which illustrate the applicability of Theorem 3.

Throughout the paper, we use $\text{conv}\, K$, $\overline{\text{conv}}\, K$, $\text{cone}\, K$, and $\text{span}\, K$ to refer to the convex hull, closed convex hull, conical hull, and linear span of a set K, respectively. We also use $\text{bd}\, K$, $\text{int}\, K$, and $\dim K$ to refer the boundary, interior, and dimension of K. The *dual cone* of $K \subseteq \mathbb{R}^n$ is $K^* := \{\alpha \in \mathbb{R}^n : x^\top \alpha \ge 0 \ \forall x \in K\}$. The second-order cone $\mathbb{L}^n$ is self-dual, that is, $(\mathbb{L}^n)^* = \mathbb{L}^n$. Given a vector $u \in \mathbb{R}^n$, we let $\tilde{u} := (u_1; \dots; u_{n-1})$ denote the subvector obtained by dropping its last entry.

2. Intersection of the second-order cone with an affine subspace

In this section, we show that the continuous relaxation C can be assumed to be the intersection of a lower-dimensional second-order cone with a single hyperplane. Let $E := \{x \in \mathbb{R}^n : Ax = b\}$ so that $C = \mathbb{L}^n \cap E$. We are going to use the following lemma to simplify our analysis.

Lemma 1. *Let V be a p-dimensional linear subspace of $\mathbb{R}^n$. The intersection $\mathbb{L}^n \cap V$ is either the origin, a half-line, or a bijective linear transformation of $\mathbb{L}^p$.*

See Section 2.1 of [5] for a similar result. We do not give a formal proof of Lemma 1 but just note that it can be obtained by observing that the second-order cone is the conic hull of a (one dimension smaller) sphere, and that the intersection of a sphere with an affine space is either empty, a single point (when the affine space intersects the sphere but not its interior), or a lower dimensional sphere of the same dimension as the affine space (when the affine space intersects the interior of the sphere).

Lemma 1 implies that, when $b = 0$, C is either the origin, a half-line, or a bijective linear transformation of $\mathbb{L}^{n-m}$. The closed convex hull $\overline{\text{conv}}(C_1 \cup C_2)$ can be described easily when C is a single point or a half-line. Furthermore, the problem of characterizing $\overline{\text{conv}}(C_1 \cup C_2)$ when C is a bijective linear transformation of $\mathbb{L}^{n-m}$ can be reduced to that of convexifying an associated two-term disjunction on $\mathbb{L}^{n-m}$. We refer the reader to [14] for a detailed study of the closed convex hulls of two-term disjunctions on the second-order cone.

In the remainder, we focus on the case $b \ne 0$. Note that, whenever this is the case, we can permute and normalize the rows of (A, b) so that its last row reads $(a_m^\top, 1)$, and subtracting a multiple of $(a_m^\top, 1)$ from the other rows if necessary, we can write the remaining rows of (A, b) as $(\tilde{A}, 0)$. Therefore, we can assume without any loss of generality that all components of b are zero except the last one. Isolating the last row of (A, b) from the others, we can then write

$$E = \left\{x \in \mathbb{R}^n : \tilde{A}x = 0, \ a_m^\top x = 1\right\}.$$

Let $V := \{x \in \mathbb{R}^n : \tilde{A}x = 0\}$. By Lemma 1, $\mathbb{L}^n \cap V$ is the origin, a half-line, or a bijective linear transformation of $\mathbb{L}^{n-m+1}$. Again, the first two cases are easy and not of interest in our analysis. In the last case, we can find a matrix D whose columns form an orthonormal basis for V and define a nonsingular matrix H such that $\{y \in \mathbb{R}^{n-m+1} : Dy \in \mathbb{L}^n\} = H\mathbb{L}^{n-m+1}$. Then we can represent C equivalently as

$$\begin{aligned} C &= \left\{x \in \mathbb{L}^n : x = Dy, \ a_m^\top x = 1\right\} \\ &= D\left\{y \in \mathbb{R}^{n-m+1} : Dy \in \mathbb{L}^n, \ a_m^\top Dy = 1\right\} \\ &= D\left\{y \in \mathbb{R}^{n-m+1} : y \in H\mathbb{L}^{n-m+1}, \ a_m^\top Dy = 1\right\} \\ &= DH\left\{z \in \mathbb{L}^{n-m+1} : \ a_m^\top DHz = 1\right\}. \end{aligned}$$

The set $C = \mathbb{L}^n \cap E$ is a bijective linear transformation of $\{z \in \mathbb{L}^{n-m+1} : a_m^\top DHz = 1\}$. Furthermore, the same linear transformation maps any two-term disjunction in $\{z \in \mathbb{L}^{n-m+1} : a_m^\top DHz = 1\}$ to a two-term disjunction in C and vice versa. Thus, without any loss of generality, we can take $m = 1$ in (1) and study the problem of describing $\overline{\text{conv}}(C_1 \cup C_2)$ where

$$\begin{aligned} C &= \left\{x \in \mathbb{L}^n : a^\top x = 1\right\}, \\ C_1 &= \left\{x \in C : l_1^\top x \ge l_{1,0}\right\}, \quad \text{and} \\ C_2 &= \left\{x \in C : l_2^\top x \ge l_{2,0}\right\}. \end{aligned} \qquad (2)$$

In Section 4 we will give a full description of $\overline{\text{conv}}(C_1 \cup C_2)$ under certain conditions.

3. Homogeneous two-term disjunctions on the second-order cone

In this section, we study the convex hull of a homogeneous two-term disjunction $c_1^\top x \ge 0 \vee c_2^\top x \ge 0$ on the second-order cone. Let

$$Q_1 := \left\{x \in \mathbb{L}^n : c_1^\top x \ge 0\right\} \quad \text{and} \quad Q_2 := \left\{x \in \mathbb{L}^n : c_2^\top x \ge 0\right\}. \qquad (3)$$

The main result of this section characterizes $\text{conv}(Q_1 \cup Q_2)$. Note that Q_1 and Q_2 are closed, convex, pointed cones; therefore, $\text{conv}(Q_1 \cup Q_2)$ is always closed (see, e.g., Rockafellar [17, Corollary 9.1.3]).

When $Q_1 \subseteq Q_2$, we have $\text{conv}(Q_1 \cup Q_2) = Q_2$. Similarly, when $Q_1 \supseteq Q_2$, we have $\text{conv}(Q_1 \cup Q_2) = Q_1$. In the remainder of this section, we focus on the case where $Q_1 \not\subseteq Q_2$ and $Q_1 \not\supseteq Q_2$.

Assumption 1. $Q_1 \not\subseteq Q_2$ and $Q_1 \not\supseteq Q_2$.

We also make the following technical assumption.

Assumption 2. $Q_1 \cap \text{int}\, \mathbb{L}^n \ne \emptyset$ and $Q_2 \cap \text{int}\, \mathbb{L}^n \ne \emptyset$.