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A closed-form expansion approach for pricing discretely monitored variance swaps

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ABSTRACT

Variance swaps are among the most useful tools for derivatives trading and risk management. For pricing discretely monitored variance swaps under a general class of jump–diffusion models, we propose a closed-form expansion based on the length of monitoring interval. Our method relies on an iterative application of the Dynkin formula, which is usually called the operator method in financial econometrics. Numerical examples are given for demonstrating the efficiency of the method.

1. Introduction

A variance swap is a financial derivative that allows one to speculate on or hedge risks associated with the volatility of some underlying assets like a stock index, exchange rate, or interest rate. One leg of the swap pays an amount based on the realized variance of the return of the underlying asset. The other leg of the swap pays a fixed amount, which is the fair strike and is determined so that the arbitrage-free price of the swap is zero at the inception of the contract. See, for example, [12,11], and Chapter 11 [16] for details.

Conventionally, the realized variance is usually calculated as an annualized total sum of squared log returns. The theory of quadratic variation guarantees that such a notion based on discretely monitored paths of the underlying asset price converges to an annualized integral of variance, a continuous approximation of realized variance, which has been extensively applied by most studies in pricing or measuring volatility risk owing to its mathematical convenience, see, for example, the survey and discussions in [20]. However, as pointed out in [17], when pricing variance swaps with relatively small maturities, pricing errors are usually significant, if one employs the annualized integral of variance as a proxy of the realized variance. Investigations on the differences and convergence properties between discretely monitored realized variance and its continuous counterpart can be found in, for example, [8,20]. Among others, this finding motivates the challenging study of pricing discretely monitored variance swaps directly using annualized total sum of squared log returns. For affine stochastic volatility (with jump) models (see, for example, [13]), various analytical methods for pricing related derivatives can be found in, for example, [8,34,33,30].

In this paper, we propose a closed-form expansion method for pricing discretely monitored variance swaps in an arbitrary class of stochastic volatility with jump models with flexible specifications without particular assumptions like affine structures. Thus, our method broadens the scope of analytical pricing methods to a wider range of models. The expansion is convenient to implement symbolically in any symbolic softwares (e.g., Mathematica) and can be saved as pricing formulas for numerical calculations, which can be done almost instantaneously. Thus, comparing with Monte Carlo simulations, a significant reduction of CPU computing time can be achieved.

Our expansion method roots in iterative applications of the celebrated Dynkin formula for Markov diffusion models, see, for example, Chapter 7 in [29]. As a stochastic generalization of the fundamental theorem of classical calculus, the Dynkin formula centralizes the theory and application of stochastic calculus. It renders the expected value of smooth functionals of a diffusion at a stopping time. Iterative applications of this formula (iterated Dynkin formula hereafter) result in a stochastic analogy of the

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Taylor expansion in classical calculus. Thus, it provides us with a tool for developing a closed-form series expansion based on the length of the monitoring interval of the variance swaps. As shown momentarily, mathematical treatment is needed to handle the path-dependency nature of the variance swap. Known as the operator method, iterated Dynkin formulas have established their important roles in financial econometrics, see, for example, [21,1,2], as well as [3] and the references therein. Our expansion can be regarded as a small-time type expansion for smooth Wiener functionals. Relying on the theory of [31,32], applications of small-time expansions to non-smooth generalized Wiener functionals can be found in, for example, [23,25,10].

The rest of the paper is organized as follows. In Section 2, we set up the definition of discretely monitored variance swaps and the model. In Section 3, we propose the closed-form expansion method. For the purpose of illustration, numerical experiments are given in Section 4. We conclude the paper in Section 5. As an online supplementary material (see Appendix A), Li and Li [26] collects more details on the computational results, an illustration of expansion formulas, and an interpretation of our expansions.

2. The model and basic setup

2.1. Variance swaps

Assume that the asset prices are observed discretely at $t_j = j\Delta$, where $j = 0, 1, 2, \ldots$, and Δ represents the length of the monitoring interval of the variance swaps, for example, $\Delta = 1/252$, $1/52$, and $1/12$ for daily, weekly, and monthly, respectively. For a time horizon with m such periods, the realized variance over $[0, m\Delta]$ is defined as

$$RV_{m,\Delta} = \frac{1}{m\Delta}\sum_{j=1}^{m}\left(\log\frac{S_{t_j}}{S_{t_{j-1}}}\right)^2.$$

Then, the payoff of a variance swap with maturity $T = m\Delta$ is defined as $(RV_{m,\Delta} - K^*) \times N$, where K^* is the fair strike making the initial arbitrage-free price of this swap zero and N is the notional amount of the swap, see, for example, explanations in [9]. Thus, we have

$$K^* = E[RV_{m,\Delta}] = \frac{1}{m\Delta}\sum_{j=1}^{m} E\left(\log\frac{S_{t_j}}{S_{t_{j-1}}}\right)^2, \tag{2.1}$$

where the expectations are calculated under the risk-neutral probability measure for derivatives pricing.

2.2. The model

Under the risk-neutral probability measure, we assume that the price of an asset is governed by the following stochastic differential equation (SDE hereafter):

$$\frac{dS_t}{S_t} = (r(X_t) - d(X_t) - \lambda\mu t)dt + \sum_{i=1}^{d}\sigma_i(X_t)dW_t^{(i)} + dJ_t,$$
$$S_0 = s_0. \tag{2.2}$$

Here, X is an n dimensional diffusion governed by

$$dX_t = \alpha(X_t)dt + \beta(X_t)dW_t, \quad X_0 = x_0, \tag{2.3}$$

where $W = (W^{(1)}, W^{(2)}, \ldots, W^{(d)})^\top$ is a d dimensional standard Brownian motion; $r(x)$, $d(x)$, and $\sigma_i(x)$, $i = 1, 2, \ldots, d$, are functions from R^n to R; $\alpha(x) = (\alpha_1(x), \alpha_2(x), \ldots, \alpha_n(x))^\top$ is an n dimensional vector-valued function and $\beta = (\beta_{ij}(x))_{n\times d}$ is an $n \times d$ matrix-valued function. Here, $r(X_t)$ and $d(X_t)$ represent the spot interest rate and the dividend rate, respectively. J is an independent compound Poisson process defined by $J_t = \sum_{k=1}^{N_t}(\exp(Z_k) - 1)$, where N is a Poisson process with intensity λ; Z_k's are identically independently distributed random variables representing jump sizes with $\mu = E(\exp(Z_k) - 1)$.

This model nests many sophisticated models proposed in the literature of derivatives pricing, for example, stochastic volatility models in [19,18,5], multifactor stochastic volatility models in [6,13,17,14], jump–diffusion models in [27,22], stochastic volatility with jump models in [6,13,7,4].

3. A closed-form expansion approach for pricing variance swaps

3.1. The operator method

Our closed-form expansion hinges on the following stochastic analogy of the Taylor expansion. Suppose that ξ is an r dimensional diffusion model governed by the following SDE:

$$d\xi_t = a(\xi_t)dt + b(\xi_t)dW_t. \tag{3.1}$$

For a smooth enough function f taking values in R^r and any $t > 0$, the expectation $Ef(\xi_t)$ admits the following Taylor-like formula

$$Ef(\xi_t) = \sum_{k=0}^{K}\mathcal{L}^k f(\xi_0)\frac{t^k}{k!} + R_{K+1}, \tag{3.2}$$

for any arbitrary order K, where $\mathcal{L}$ is the infinitesimal generator of (3.1), i.e.,

$$\mathcal{L} = \sum_{i=1}^{r} a_i(\xi)\frac{\partial}{\partial\xi_i} + \frac{1}{2}\sum_{i,j=1}^{r}(b(\xi)b^\top(\xi))_{i,j}\frac{\partial^2}{\partial\xi_i\partial\xi_j}, \tag{3.3}$$

with $(b(\xi)b^\top(\xi))_{i,j}$ referring to the i,jth element of the matrix $b(\xi)b^\top(\xi)$, and R_{K+1} is the remainder term taking the following "differential form":

$$R_{K+1} = \frac{t^{K+1}}{(K+1)!}E\left[\mathcal{L}^{K+1}f(\xi_s)\right], \tag{3.4}$$

for some $0 \le s \le t$, or "integral form":

$$R_{K+1} = \int_0^t\int_0^{s_1}\cdots\int_0^{s_K} E\mathcal{L}^{K+1}f(\xi_{s_{K+1}})ds_{K+1}\cdots ds_2 ds_1. \tag{3.5}$$

Because this formula could be easily proved by iterative applications of the Dynkin formula (see, for example, Chapter 7 in [29]), it is often referred to as an iterated Dynkin formula, which has been widely applied as a tool in stochastic analysis and various applications. As shown momentarily in the next section, we focus on the case of f being a polynomial in the underlying diffusion ξ. Subject to some technical sufficient conditions, the remainder term R_{K+1} in (3.2) can be understood as $o(t^K)$ as $t \to 0$. For example, if, for any integer $k > 0$, any arbitrary kth order derivative in ξ of the functions $a(\xi)$ and $b(\xi)$ exists and is bounded for any ξ in the state space of the process ξ, there exists a positive constant C_K such that

$$|R_{K+1}| < C_K t^{K+1}. \tag{3.6}$$

The proof of this claim follows from standard tools for analyzing stochastic differential equations, see, for example, the arguments in Section 2.2 of [28] and Lemma 2 in [24] as well as Theorem 2.2 in [31]. Owing to the length limit of this paper, we omit the proof. Furthermore, theoretical relaxation of the aforementioned sufficient conditions for (3.6) can be regarded as a future research topic.

Thus, the formula (3.2) can be regarded as a stochastic analogy of the celebrated Taylor expansion in classical calculus, which renders a local property (as $t \to 0$) rather than giving a full investigation on the radius of convergence. Thus, the expansion $\sum_{k=0}^{K}\mathcal{L}^k$