Operations Research Letters 43 (2015) 545–549

Contents lists available at ScienceDirect

Operations Research Letters

journal homepage: www.elsevier.com/locate/orl

Turnpike properties of group testing for incomplete identification

Michael Zhang, Jiejian Feng *

Sobey School of Business, Saint Mary's University, Nova Scotia, B3H 3C3, Canada

ARTICLE INFO

Article history:
Received 9 November 2014
Received in revised form
5 August 2015
Accepted 13 August 2015
Available online 20 August 2015

Keywords:
Group testing
Incomplete identification
Turnpike

ABSTRACT

A manufacturer needs to incompletely identify plenty of electronics chips with 100% quality from group testable chips. The computation workload of finding the optimal dynamic testing policy is terrible by the bottom-up approach. We prove that turnpike properties exist, i.e. when the problem scale is large enough, the optimal testing size is always the same as the test size of maximizing the number of expected perfect chips in one test, and it depends on only the defective rate of chips.

1. Introduction

Consider the following decision problem faced by an electronics firm which needs a certain (fixed) number of chips of 100% quality required for the assembly of a "mission critical" component. The firm can purchase a batch of chips from a supplier, which is with a known defective rate, at a very low unit price (e.g., \$0.05/unit), and then search for 100% quality chips from this batch of chips. To achieve the minimum expected cost, the firm can solve a two-stage decision problem: (i) What is the optimal number of less-than 100% quality chips to purchase, (ii) What is the optimal testing scheme to incomplete identify some 100% quality chips among the less-than-100% batch. Assume that the firm can purchase any number of 100% quality chips at a high unit price (e.g., \$2.5/unit) if the number of 100% quality chips is less than the demand at the end of the testing process.

The group testing for incomplete identification is that certain number of less-than 100% quality chips are simultaneously tested in a group. The test result will be success if each chip in the group is 100% quality; otherwise, (i.e. at least one chip in the group is defective,) the test result will be failure, and the whole group may be discarded because it is not the testing target to figure out all the defective chips. This is different from the group testing for complete identification that is to identify the quality of each individual at last such as HIV tests for people.

A number of researchers have studied the firm's decision problem with incomplete identification. Bar-Lev, Boneh, and Perry [1] found out the optimal number of contaminated chips and the optimal testing size when the group testing size was assumed to be a constant. Bar-Lev, Parlar and Perry [2] formulated a stochastic dynamic program to determine the optimal set of dynamic group testing sizes, each of which related to one state (x, y) where suppose x is the number of the latest unsatisfied demands, and y is the number of the latest untested contaminated chips. Then, the expected cost information at some states was used to determine the optimal number of contaminated chips. They showed that the dynamic approach could significantly reduce the expected cost although they had not provided efficient technology on computation. When the problem scale becomes large, the computation workload can be very heavy. For this, Feng, Liu and Parlar [5] built the bounds of the optimal testing size, and kept on decreasing the bounds of the optimal number of contaminated chips in the searching process. Thus, it is possible to solve practice problems at relative large scale although the optimal group testing size varies in haphazard order between bounds.

One remained problem is in the asymptotic situation, i.e. the problem scale is extremely large. As we intuitively expect, the process of searching for the optimal policy would be very complex and time consuming if the haphazard order of the optimal group testing size went on forever; however, the problem would suddenly become very simple out of expectation if the optimal group testing size followed a simple law such as approaching to a constant, i.e. having turnpike properties. An idea close to the turnpike property was introduced by John von Neumann [7]. The turnpike phenomenon was first observed by Paul A. Samuelson in 1948 as Zaslavski [9] stated; then, the turnpike theorem was widely known from Dorfman, Samuelson, and Solow [4].

* Correspondence to: SB103, 923 Robie Street, Halifax, Nova Scotia, B3H 3C3, Canada. Tel.: +1 902 4893058.

E-mail addresses: michael.zhang@smu.ca (M. Zhang), jiejian.feng@smu.ca (J. Feng).

http://dx.doi.org/10.1016/j.orl.2015.08.004
0167-6377/

McKenzie [6] reviewed the turnpike theory, and pointed out two characteristics of traditional turnpike theorems. Bewley [3] proved a turnpike theorem of a general equilibrium model. Yano [8] provided a standard to compare with realistic models. Trélat and Zuazua [10] studied the turnpike property for a dynamical and nonlinear optimal control problem. The turnpike theorem has been used to discuss efficient accumulation, which looks like traveling from one location to another location through part of turnpikes in a traffic net, in an economic system.

Specially, for the group testing of chips, the existence of turnpike property is a misgiving. One reviewer of the paper Feng, Liu and Parlar [5] leaded the authors to consider this interesting problem where a turnpike is the optimal, simple, and easy procedure in practice instead of a policy with the shortest testing time. The authors translated it to a simpler problem, but they had not determined the existence of the turnpike properties. This paper provides a positive answer on the existence. The turnpike theorem belongs to the second kind, which was called the early turnpike that optimal paths stayed within (a small neighborhood of) the turnpike in the initial phase, in McKenzie [6]. As a result, when the problem scale is large, the firm can firstly test with a fixed group testing size, and switch back to dynamic policy when the scale is reduced to certain level so the minimum cost will be achieved.

2. Dynamic optimization model

Note that x units of certain 100% quality (i.e. perfect) chips are required for the assembly of products. The perfect chips can be purchased at a premium price of \$$\pi$ per unit; it is also possible to purchase a batch of the same type of contaminated chips at a much lower price of \$$c$ per unit ($c \ll \pi$) as each chip in the batch is independently perfect with probability p and defective with probability $1-p$. The contaminated chips are group-testable as the following: the group size of a test can be any number, and the outcome of a group test can be either success or failure, but not both. The success outcome indicates that all the chips in the tested group are perfect; the failure outcome indicates that *at least one* chip in the tested group is defective. Here, a group with failure outcome will be abandoned without further testing to know which chips (or how many chips) are defective. Testing fee is \$$K$ each time no matter testing size.

Assume that y units of the contaminated chips have been purchased. To satisfy the demand of x perfect chips with the minimum expected cost, the next problem is to determine a testing procedure for finding out some perfect units from those available contaminated chips. At the end of testing procedure, either the demand has been satisfied or less than x perfect chips have been found. For the second result, the shortfall will be covered by purchasing perfect units.

When x and y are not large, the optimal testing size $u(x, y)$ at each state (x, y) can be efficiently determined by the bottom-up approach, shown as Bar-Lev, Parlar and Perry [2], and Feng, Liu and Parlar [5]. The objective of this paper is to study the turnpike properties of the group-testing problem at large x and y.

The expected number of perfect units from a group test of u units is given by $\phi(u) = up^u + 0(1-p^u) = up^u$ for $u \geq 0$. $\phi(u)$ is maximized by choosing a group size of $\hat{u} = -1/\ln(p)$. (See Lemma 1 in the Appendix.) When $\phi(\hat{u}) > K/\pi$, the nonlinear equation $\phi(u) = K/\pi$ will have two distinct roots at u' and u'' with $u' < u''$. Similarly, when $\phi(\hat{u}) = K/\pi$, the equation will have exactly one root at $\hat{u}$, and when $\phi(\hat{u}) < K/\pi$, the equation will have no roots. Since the test size has to be an integer and $-1/\ln(p)$ may not be an integer, without lose of generality, assume that $\hat{u}p^{\hat{u}}$ is larger at $\hat{u} = \lfloor -1/\ln(p) \rfloor$ than $\hat{u} = \lceil -1/\ln(p) \rceil$ to simplify the following presentation. In other words, let $\hat{u} = \lfloor -1/\ln(p) \rfloor$.

Let $f(x, y)$ be the minimum of expected cost when the state is (x, y) and $\hat{f}(x)$ be the minimum of expected cost when the number of required perfect items is x and there are infinite many group testable items. Feng, Liu and Parlar [5] showed that

(1) when y is finite,

$$f(x,y) = \min_{u' \leq u \leq \min(x,y,u'')} \big\{\pi x, K + p^u f(x-u, y-u) + (1-p^u) f(x, y-u)\big\}$$

with the simpler boundary condition

$$f(x,y) = \begin{cases} \pi x, & \text{if } (x,y) \in \{(x,y) : 0 < x \text{ and } 0 \leq y \leq K/\pi\} \cup \{(x,y) : 0 < x \leq K/\pi \text{ and } 0 \leq y\}, \\ 0, & \text{if } (x,y) \in \{(x,y) :\ x = 0 \text{ and } 0 \leq y\}; \end{cases}$$

(2) when y is infinite,

$$\hat{f}(x) = \min_{u' \leq u \leq \min(x,u'')} \{\pi x, K + p^u \hat{f}(x-u) + (1-p^u)\hat{f}(x)\}$$

with the boundary condition

$$\hat{f}(x) = \begin{cases} \pi x, & \text{if } x \in \{1, \ldots, (\lceil u' \rceil - 1)\}, \\ 0, & \text{if } x = 0. \end{cases}$$

To study the turnpike properties of $f(x, y)$ and $\hat{f}(x)$, we define the turnpike property at different levels as the following:

Definition 1 (*x-Dependent Turnpike Property*)**.** For a given x, and sufficiently large but finite y, the function $f(x, y)$ has the x-dependent turnpike property if there exists a threshold $\bar{y}(x)$ such that the optimal testing size $u^*(x, y)$ is $\hat{u}^*(x)$ when $y > \bar{y}(x)$ [where $\hat{u}^*(x)$ is the optimal testing size of the value function $\hat{f}(x)$, independent from the number of remaining group-testable units y]. □

Definition 2 (*Turnpike Property*)**.** Given π, K and p, consider an integer $\bar{u}$ and a constant threshold level x_T. For any $x > x_T$, if there exists an integer $y_T(x)$ such that $u^*(x, y) = \bar{u}$ when $y > y_T(x)$, the value function $f(x, y)$ has the turnpike property for the optimal testing size at level x. Note that $\bar{u}$ does not depend on x, but $y_T(x)$ still depends on x. □

Definition 3 (*Turnpike Property with Infinite Group Testable Items*)**.** Given π, K and p, consider an integer $\bar{u}$ and a constant threshold level x_T. For any $x > x_T$, if $u^*(x) = \bar{u}$, then the value function $\hat{f}(x)$ has the turnpike property of the optimal testing size. □

A more general turnpike property for $f(x, y)$ and $\hat{f}(x)$ where the testing size totally does not depend on x can be defined as follows:

Definition 4 (*Super Turnpike Property*)**.** Given π, K and p, consider an integer u_0, a constant threshold level x_T on required perfect chips, and a constant threshold level y_T on available contaminated chips. (Here, y_T is independent of x.) The function $f(x, y)$ has the super turnpike property of the optimal testing size if $u^*(x, y) = u_0$ for any $x > x_T$ and $y > y_T$. □

3. Turnpike properties

In Feng, Liu and Parlar [5], Proposition 3 showed that the function $f(x, y)$ has the x-dependent turnpike property, and Corollary 2 showed that the function $f(x, y)$ has the turnpike property if and only $\hat{f}(x)$ has the turnpike property. But they did not confirm that $f(x, y)$ has the turnpike property or not. We are ready to answer this question in this section. (Note that the proofs in this paper based on some results in Feng, Liu and Parlar [5] so that we listed the referred results in the Appendix.)