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A new class of dual upper bounds for early exercisable derivatives encompassing both the additive and multiplicative bounds

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ABSTRACT

We present a new class of upper bounds for the Monte Carlo pricing of Bermudan derivatives. This class contains both the additive and multiplicative upper bounds as special cases. We also see that the hypothesis that the pay-off is positive for the multiplicative upper bound is unnecessary. The variance of these upper bounds is zero when the optimal hedge is chosen.

1. Introduction

In recent years much progress has been made on the development of methodologies for finding the prices of early exercisable derivatives by Monte Carlo simulation. For a long time, the focus was mainly on lower bounds with bundling and regression-based methods proving effective such as Carrière, [5], Tsitsiklis and Van Roy, [17], Longstaff and Schwartz, [14] proving popular and effective. However, it is very hard to be confident that a lower bound price is accurate without a corresponding upper bound for comparison. A second stream of research has therefore developed in recent years focused on the problem of how to find an upper bound. There have been two main approaches generally referred to as "additive" and "multiplicative". The purpose of this paper is to develop a new class of upper bounds which unifies these two approaches and contains them both as simple special cases.

Inspired by work of Davis and Karatzas, [8], the additive approach discovered independently by Rogers, [15], and Haugh and Kogan, [9], relies on a decomposition of the price process of the unexercised option into a martingale and an increasing process. It can be viewed as a seller's price: it is the cost of hedging the early exercisable derivative no matter when the buyer exercises even if the buyer possesses additional information. This philosophical point of view is discussed in detail in [11,12]. There have now been numerous papers on the method with the most popular implementation methodology being that of Andersen and Broadie, [1]. Their approach is to use the product itself with a sub-optimal exercise strategy as a hedge and it thus relies on sub-simulations.

The multiplicative method introduced by Jamshidian, [10], has proven less popular. The essential difference is that the unexercised option price is written as product of a martingale and an increasing process rather than a sum. The principal reason for this difference in popularity is probably that, as Chen and Glasserman, [7], observe, the variances achieved by the method are much higher. In particular, when the hedge is optimal the additive method has zero variance and the multiplicative method does not. However, Joshi and Tang showed in [13] that the use of a simple control variate renders the multiplicative method competitive. The lack of popularity is therefore no longer justified. In addition, in its original formulation the final pay-off had to be positive rather than non-negative. However, that restriction is also no longer necessary and our general result here only requires non-negativity.

In this note, we develop a new class of upper bounds that relies on the choice of two martingales. The class contains both the additive upper bound and the Joshi–Tang formulation of the multiplicative one. The key idea lies in how mismatched cash-flows are reinvested: the additive upper bound implicitly assumes the numeraire, and the multiplicative one the product itself. However, any positive product could be used and each choice leads to a different upper bound.

There have been various improvements to the original additive and multiplicative methods. The original methods require taking a maximum along a path. This maximum can occur even when the option is out of the money and so exercise would never happen there. [2,12] show that one can restrict the maximum to points in the money for $t < T$. More generally, once one has proven that

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the optimal strategy lies in a given class, one can also restrict the maximum to the same class, [4]. Whilst we do not explicitly prove these results for our new class of upper bounds, it is clear that they go over when appropriately formulated.

2. Setup and review

We establish notation. Throughout, we will work with deflated prices. We let $(Z_j),\ j = 0, \ldots, T$ be a non-negative adapted process on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. We assume for some $p,\ Z_j \in L^p$. The holder of the Bermudan derivative receives the non-negative deflated pay-off, Z_j, at time j if they exercise at time j. We shall assume that the derivative cannot be exercised at time zero, and must be exercised at time T if it has not been previously exercised. This simplifies the mathematics in that we do not have to deal with the case of non-exercise and an exercise out of the money at maturity simply yields a zero cash flow.

We will work in a fixed martingale measure throughout. We assume that a numeraire has been fixed and that all prices have been deflated by it. Let D_t be the deflated price of the derivative at time t if it has not been previously exercised.

The Rogers/Haugh–Kogan additive method, [15,9], requires us to find a martingale M_t. The upper bound is then given as

$$D_0 \leq M_0 + \mathbb{E}(\max_j (Z_j - M_j)).$$

The inventors of the method showed that if M_j is chosen to be the deflated price process of the product itself with the optimal exercise strategy and correct reinvestment strategy the upper bound is the true price. The reinvestment strategy is that at time of exercise, one must "buy" the product with one less exercise date using the exercise value. Since the strategy is optimal, this will always leave a positive surplus which is used to buy units of the numeraire, whose deflated value is necessarily constant. See [10] for this interpretation. The authors did not view the method in this fashion, however, and instead defined M_j to be the martingale part of the additive Doob–Meyer decomposition of the value of D_t. Thus

$$M_t = D_t + B_t^*$$

with B_t^* non-negative and increasing.

Jamshidian's multiplicative method can be viewed in terms of a multiplicative Doob–Meyer decomposition. The surplus cash on exercise is used to buy more units of the original product rather than the numeraire. In addition, all pay-offs are evaluated in terms of how much product they will buy rather than being converted to numeraire bonds. If X_t is a positive martingale, this leads to an upper bound

$$D_0 \leq \mathbb{E}\left(\max_t \left(\left(\frac{Z_t}{X_t}\right) X_T\right)\right).$$

The optimal X_t which yields equality is of the form

$$X_t = D_t A_t^*$$

with $A_t^* \geq 1$ and increasing. Our purpose in the rest of this paper is to develop a method containing both of these as special cases.

3. Upper bounds

Let $\mathcal{M}$ denote the class of martingales with respect to $\{\mathcal{F}_t\}$. Let $\mathcal{X}$ denote the subset of $\mathcal{M}$ consisting of processes which are positive for $t < T$ and non-negative for $t = T$. Let D_t denote the deflated value of the derivative at time t assuming non-exercise before t. Note that D_t is not a martingale. In what follows τ will always denote a stopping time.

Our main result is

Theorem 1. *Let $X \in \mathcal{X}$ then*

$$\sup_\tau \mathbb{E}(Z_\tau) = \inf_{M \in \mathcal{M}} \left(M_0 + \mathbb{E}\left(\max_t \left((Z_t - M_t)\frac{X_T}{X_t}\right)\right)\right)$$

where X_T/X_T is taken to be 1 if $X_T = 0$. If $M_t = A_t D_t + B_t$ for any A_t, B_t with $A_0 = 1,\ A_t \geq 1$ and $B_0 = 0,\ B_t \geq 0$ then the infimum is attained and the equality is almost sure. Such processes M_t exist and it is possible to take $A_t = 1, \forall t$, with B_t increasing, or $B_t = 0, \forall t$, and A_t increasing.

This theorem with $\leq$ will trivially hold for any subset of $\mathcal{M}$. If the subset contains one of the optimal martingales, $A_t D_t + B_t$, then equality is achieved. Thus we could replace $\mathcal{M}$ with $\mathcal{X}$ if we so desired. Note that we have not required monotonicity of A_t and B_t. This theorem is a little asymmetric in that we optimize over M_t whilst fixing X_t. Could we do the opposite? The argument of [10] would apply and make this work if and only if we are guaranteed the positivity of $Z_T - M_T$. Such a condition appears unnatural so we do not pursue that approach.

3.1. Additive as special case

First, to see that the additive result is a special case of this, set $X_t = 1$ for all t. We then have

$$\sup_\tau \mathbb{E}(Z_t) = \inf_{M \in \mathcal{M}} \left(M_0 + \mathbb{E}\left(\max_t (Z_t - M_t)\right)\right)$$

with almost sure equality for the optimal choice of $M_t = D_t + B_t^*$. This is precisely the additive dual.

3.2. Multiplicative as special case

Relating this result to the multiplicative dual is more interesting. Of course, Jamshidian's original upper bound did not attain almost sureness even for the optimal bound so this is not identical. However, Joshi and Tang, [13] show that the same upper bound can be obtained with zero variance by the use of an appropriate control and it is that upper bound that we replicate. Using Theorem 1, we can take $A_t = A_t^* \geq 1,\ B_t = 0$ and set $M_t = X_t = A_t D_t$, and we obtain equality with X_t in the Jamshidian form, and $X_t \in \mathcal{X}$. Since $\mathcal{X} \subset \mathcal{M}$, an infimum over it must be at least as large as one over $\mathcal{M}$, we therefore have, using the fact that equality is obtained,

$$\sup_\tau \mathbb{E}(Z_t) = \inf_{X \in \mathcal{X}} \left(X_0 + \mathbb{E}\left(\max_t (Z_t - X_t)\frac{X_T}{X_t}\right)\right), \tag{3.1}$$

$$= \inf_{X \in \mathcal{X}} \left(X_0 + \mathbb{E}\left(\max_t \left(\frac{Z_t}{X_t} - 1\right) X_T\right)\right), \tag{3.2}$$

$$= \inf_{X \in \mathcal{X}} \left(X_0 + \mathbb{E}\left(\max_t \left(\left(\frac{Z_t}{X_t}\right) X_T\right) - X_T\right)\right), \tag{3.3}$$

$$= \inf_{X \in \mathcal{X}} \mathbb{E}\left(\max_t \left(\left(\frac{Z_t}{X_t}\right) X_T\right)\right). \tag{3.4}$$

The second last expression is the Joshi–Tang result and the final one is Jamshidian's result. The passing of X_T through the expectation at the final line destroys the almost sureness even when X_T is optimal. This is actually a little better than Jamshidian's result in that his derivation required $X_T > 0$. An extension to the case where $X_T = 0$ with positive probability was included in [12] under certain additional assumptions. We will see that those assumptions are unnecessary.

3.3. Financial intuition

Before proceeding to a formal proof of Theorem 1, we discuss it in financial terms. See [11,12] for more background. An alternate