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Split cuts and extended formulations for Mixed Integer Conic Quadratic Programming

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ABSTRACT

We study split cuts and extended formulations for Mixed Integer Conic Quadratic Programming (MICQP) and their relation to Conic Mixed Integer Rounding (CMIR) cuts. We show that CMIR is a linear split cut for the polyhedral portion of an extended formulation of a quadratic set and it can be weaker than the nonlinear split cut of the same quadratic set. However, we also show that families of CMIRs can be significantly stronger than the associated family of nonlinear split cuts.

1. Introduction

Split cuts [12], Gomory Mixed Integer (GMI) cuts [17], and Mixed Integer Rounding (MIR) cuts [23,24] are some of the most effective valid inequalities for Mixed Integer Linear Programming (MILP) [8]. While they are known to be equivalent [15,24], each of them provides different advantages and insights. In particular, the split cuts construction shows that they are a particular case of disjunctive cuts [3] and hence have a straightforward extension to Mixed Integer Nonlinear Programming (MINLP). The study of split cuts for MINLP is still much more limited than for MILP; however, there has been significant work on the computational use of split cuts in MINLP [9,11,16,18,25] and a recent surge of theoretical developments [1,2,4,5,14,19,21,22]. In particular, several formulas for split cuts for Mixed Integer Conic Quadratic Programming (MICQP) have been recently developed [1,4,5,14,21]. While the resulting cuts are strong nonlinear inequalities, adding these cuts to the continuous relaxation of the MICQP can significantly increase its solution time, which could negate the effectiveness of the cuts. One potential solution is to use linearizations of these cuts [9,18], but in this case there is a strong trade-off between their strength and the computational burden of generating them. An alternative approach was introduced by Atamtürk and Narayanan [2] who use the polyhedral portion of a nonlinear *extended formulation* (i.e., a formulation with auxiliary variables) to construct an inexpensive, but potentially strong, linear cut they denote the Conic MIR (CMIR). In this paper we attempt to broaden our understanding of split cuts for MINLP by providing a precise link between the CMIRs and split cuts for quadratic sets. In particular, this link provides a possible solution to the trade-off between the strength and computational burden resulting from adding the cuts to the relaxation.

Our first contribution is to show that the CMIR is a linear split cut for the polyhedral portion of the nonlinear extended formulation from [2]. Through this equivalence, we can extend the most general version of the CMIRs to the case of variables with unrestricted signs which was not previously possible. Our second contribution is to give a precise relation between the CMIR and nonlinear split cuts for quadratic sets. In particular, we show that, since the CMIR construction does not consider any quadratic information, a single CMIR can be weaker than a single nonlinear split cut. However, we also show that when families of split cuts and CMIRs are considered, CMIRs can provide a significant advantage over nonlinear split cuts by exploiting their common extended formulation. To the best of our knowledge, this is the first illustration of how the power of an extended formulation can improve the strength of a cutting plane procedure in MINLP.

The rest of the paper is structured as follows. In Section 2 we introduce some notation and describe previous results on CMIRs and split cuts for MINLP. In Section 3 we establish the equivalency

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between CMIRs and linear split cuts for an extended formulation. Finally, in Section 4 we compare the strength of nonlinear split cuts and CMIRs.

2. Notation and previous work

We let $e^i \in \mathbb{R}^n$ and $I \in \mathbb{R}^{n\times n}$ denote the ith unit vector and the identity matrix where we omit dimension n if evident from the context. We also let $\|x\|_2 := \sqrt{\sum_{i=1}^n x_i^2}$ denote the Euclidean norm of $x \in \mathbb{R}^n$ and $|x| \in \mathbb{R}^n$ be the vector whose components are the absolute value of the components of $x \in \mathbb{R}^n$. In addition, for $a \in \mathbb{R}$ we let $(a)^+ := \max\{0, a\}$ and $\lfloor a \rfloor := \max\{k \in \mathbb{Z} : k \le a\}$, and we let $[n] := \{1, \dots, n\}$. Finally, for notational convenience, we define split cuts while identifying a single set of integer variables $x \in \mathbb{Z}^n$ and three sets of continuous variables $y \in \mathbb{R}^p,\ t \in \mathbb{R}^m$, and $t_0 \in \mathbb{R}$.

Definition 1. Let $K \subseteq \mathbb{R}^{n+p+m+1}$ be a closed convex set and $(\pi, \pi_0) \in \mathbb{Z}^n \times \mathbb{Z}$. A *split cut* for K is any valid inequality for

$$K^{\pi,\pi_0} := \operatorname{conv}\big(\{(x, y, t, t_0) \in K : \pi^T x \le \pi_0\} \cup \{(x, y, t, t_0) \in K : \pi^T x \ge \pi_0 + 1\}\big)$$

for some $(\pi, \pi_0) \in \mathbb{Z}^n \times \mathbb{Z}$. If $\pi = e^i$ for some $i \in [n]$, we refer to (π, π_0) as an *elementary disjunction* and to the obtained cuts as *elementary split cuts*.

Because $K^{\pi,\pi_0} \supseteq \operatorname{conv}(K \cap (\mathbb{Z}^n \times \mathbb{R}^{p+m+1}))$, split cuts are valid inequalities for $K \cap (\mathbb{Z}^n \times \mathbb{R}^{p+m+1})$. For MILP, where K is a rational polyhedron, K^{π,π_0} is also a polyhedron and we only need linear split cuts. In contrast, if K is a general closed convex set, K^{π,π_0} is only closed and convex [14]. However, for special classes of K, we can characterize the nonlinear split cuts that need to be added to K to obtain K^{π,π_0} [1,4,5,14,19,21]. For instance, the following proposition from [21] characterizes split cuts for conic quadratic sets of the form

$$C := \left\{(x, t_0) \in \mathbb{R}^{n+1} : \|B(x - c)\|_2 \le t_0\right\}, \tag{1}$$

where C is in fact an affine transformation of the Quadratic cone $\{(x, t_0) \in \mathbb{R}^{n+1} : \|x\|_2 \le t_0\}$.

Proposition 1. *Let $B \in \mathbb{R}^{n\times n}$ be an invertible matrix, $c \in \mathbb{R}^n$, $(\pi, \pi_0) \in \mathbb{Z}^n \times \mathbb{Z}$, and C be as defined in (1). If $\pi^T c \notin (\pi_0, \pi_0 + 1)$, then $C^{\pi,\pi_0} = C$. Otherwise, there exist $\bar{B} \in \mathbb{R}^{n\times n}$ and $\bar{c} \in \mathbb{R}^n$ such that*

$$C^{\pi,\pi_0} = \left\{(x, t_0) \in C : \left\|\bar{B}(x - c) + \bar{c}\right\|_2 \le t_0\right\}.$$

Proposition 1 shows that the single split cut for C is $\|\bar{B}(x - c) + \bar{c}\|_2 \le t_0$ which is of the same class as the inequality describing C. However, this inequality can be too expensive computationally and it can be preferable to add linear cuts instead. One way to achieve this is to add a finite number of linearizations of the nonlinear cuts. Such linearizations can be algorithmically obtained even in the absence of nonlinear cut formulas. Two examples of this are the algorithms introduced in [9,18] to generate disjunctive inequalities for convex MINLPs.

A completely different linearization scheme was introduced by Atamtürk and Narayanan [2] for the general conic quadratic set given by

$$M_+ := \left\{(x, y, t_0) \in \mathbb{R}^{n+p+1} : \|Ax + Gy - b\|_2 \le t_0,\ x \ge 0,\ y \ge 0\right\},$$

for rational matrices and vectors $A \in \mathbb{Q}^{m\times n},\ G \in \mathbb{Q}^{m\times p}$, and $b \in \mathbb{Q}^m$. Instead of considering valid inequalities for $\operatorname{conv}(M_+ \cap (\mathbb{Z}^n \times \mathbb{R}^{p+1}))$ directly, using auxiliary variables $t \in \mathbb{R}^m$, they first introduce the nonlinear extended formulation of M_+ given by

$$|Ax + Gy - b| \le t, \quad x \ge 0,\ y \ge 0,\ \ \|t\|_2 \le t_0, \tag{2}$$

so that, if $P_+ := \{(x, y, t) \in \mathbb{R}^{n+p+m} : |Ax + Gy - b| \le t, x \ge 0, y \ge 0\}$ and $\operatorname{Proj}_{(x,y,t_0)}$ is the projection onto the (x, y, t_0) space, then

$$M_+ = \operatorname{Proj}_{(x,y,t_0)}\left(\left\{(x, y, t, t_0) \in \mathbb{R}^{n+p+m+1} : \|t\|_2 \le t_0,\ (x, y, t) \in P_+\right\}\right).$$

They then exploit the fact that P_+ is a polyhedron to generate a class of valid inequalities they denote the *Conic MIR* (CMIR). The first version of the CMIR is a simple but strong cut for a four variable and one constraint version of P_+.

Proposition 2 (*Simple CMIR*)**.** *Let $b_0 \in \mathbb{R},\ f = b_0 - \lfloor b_0 \rfloor$,*

$$S_0 := \left\{(x, y, t_0) \in \mathbb{R}^4 : |x + y_1 - y_2 - b_0| \le t_0,\ y_1, y_2 \ge 0\right\},$$

and let the simple CMIR be the inequality given by

$$(1 - 2f)(x - \lfloor b_0 \rfloor) + f \le t_0 + y_1 + y_2. \tag{3}$$

The simple CMIR is valid for $\operatorname{conv}(S_0 \cap (\mathbb{Z} \times \mathbb{R}^2_+ \times \mathbb{R}_+))$ and furthermore

$$\operatorname{conv}(S_0 \cap (\mathbb{Z} \times \mathbb{R}^2_+ \times \mathbb{R}_+)) = \{(x, y, t_0) \in S_0 : (3)\}.$$

The simple CMIR is a linear inequality, but Atamtürk and Narayanan show that it can induce nonlinear inequalities in the (x, t_0) space through (2).

Lemma 1 (*Nonlinear CMIR*)**.** *Let $T_0 := \left\{(x, y, t_0) \in \mathbb{R}^3 : \sqrt{(x - b_1)^2 + y^2} \le t_0\right\}$, $P_0 := \left\{(x, y, t) \in \mathbb{R}^4 : |x - b_1| \le t_1, |y| \le t_2\right\}$, $b_1 \in \mathbb{R}$, and $f = b_1 - \lfloor b_1 \rfloor$. Then the simple CMIR for $|x - b_1| \le t_1$ is given by*

$$(1 - 2f)(x - \lfloor b_1 \rfloor) + f \le t_1, \tag{4}$$

$\operatorname{conv}(T_0 \cap (\mathbb{Z} \times \mathbb{R}^2)) = T_0^{e^1,\lfloor b_1 \rfloor}$, *and*

$$\begin{aligned} T_0^{e^1,\lfloor b_1 \rfloor} &= \left\{(x, y, t_0) \in T_0 : \sqrt{((1 - 2f)(x - \lfloor b_1 \rfloor) + f)^2 + y^2} \le t_0\right\} \\ &= \operatorname{Proj}_{(x,y,t_0)}\left(\left\{(x, y, t, t_0) \in \mathbb{R}^5 : (x, y, t) \in P_0,\ \|t\|_2 \le t_0,\ (4)\right\}\right). \end{aligned}$$

Atamtürk and Narayanan follow the traditional linear MIR procedure [23,24] to get CMIRs for M_+ and develop a super-additive version of the CMIR. Their most general version results in the following family of cuts.

Theorem 1 (*Super-Additive CMIR*)**.** *Let $a, v \in \mathbb{R}^n$, $g, w \in \mathbb{R}^p$, $h, u \in \mathbb{R}^m$, $S_+ := \left\{(x, y, t) \in \mathbb{R}^{n+p+m} : \left|a^T x + g^T y + h^T t - b_0\right| \le u^T t + v^T x + w^T y,\ x, y, t \ge 0\right\}$ be a relaxation of P_+ and let $\varphi_f(a) = -a + 2(1 - f)\left(\lfloor a \rfloor + \frac{(a - \lfloor a \rfloor - f)^+}{1 - f}\right)$. Then for any $\alpha \ne 0$ and $f_\alpha = b_0/\alpha - \lfloor b_0/\alpha \rfloor$, a valid cut for S_+ and P_+ is*

$$\sum_{j=1}^{n} \varphi_{f_\alpha}(a_j/\alpha) x_j - \varphi_{f_\alpha}(b_0/\alpha) \le \left((u + |h|)^T t + (w + |g|)^T y + v^T x\right)/|\alpha|. \tag{5}$$

We let a super-additive CMIR be any cut of this form obtained for some relaxation S_+, which can be constructed through various aggregation procedures. Finally, with regard to its relation to the traditional linear MIR, Atamtürk and Narayanan use the aggregation to show that every MIR is a CMIR. In Section 3 we show that these two cuts are in fact equivalent.

3. Conic MIR and linear split cuts

We now show that CMIRs are equivalent to linear split cuts for P_+, which are in turn equivalent to traditional linear MIRs for P_+.