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Totally unimodular multistage stochastic programs

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ABSTRACT

We consider totally unimodular multistage stochastic programs, that is, multistage stochastic programs whose extensive-form constraint matrices are totally unimodular. We establish several sufficient conditions and identify examples that have arisen in the literature.

1. Introduction

We consider a class of multistage stochastic programs (MSP) whose extensive-form constraint matrices are totally unimodular (TU). TU matrices have been well studied in deterministic mixed integer programming and combinatorial optimization. When the constraint matrix of a mixed integer program (MIP) is TU and the right-hand side is integral, the polyhedron described by the linear relaxation of the MIP is integral [7].

Kong et al. [10] provided several sufficient conditions for two-stage totally unimodular stochastic programs. These conditions used a generalization of TU matrices that we revisit in Section 2. This paper can be viewed as a multistage generalization of Kong et al. [10]. Romeijnders et al. [12] studied two-stage stochastic mixed integer programs in which the only uncertainty is on the right-hand side. They established that if the probability distribution over the right-hand side is independent and uniform and the recourse matrices are totally unimodular, a certain approximation of the integer recourse function is precisely its convex hull. Huang [8] reformulated the stochastic single item, uncapacitated dynamic lot-sizing problem without setup costs and proved that the reformulation is totally unimodular. We revisit the problem and show that the total unimodularity of Huang's [8] original formulation follows from the characterizations in Section 3.

2. Preliminaries

Consider a multistage stochastic mixed-integer program with recourse. For notational convenience and without loss of generality, we assume that the number of continuous decision variables, l, the number of integer decision variables, $n-l$, and the number of constraints, m, are the same in every stage by introducing zero row and column vectors if necessary. The deterministic equivalent program (DEP) of an MSP is

$$(DEP) \quad \min\ {c^1}^\top x^1 + \mathcal{Q}^2(x^1) \tag{1a}$$

$$\text{s.t. } W^1x^1 \geq h^1, \tag{1b}$$

$$x^1 \in \mathbb{R}^l_+ \times \mathbb{Z}^{n-l}_+,$$

where $\mathcal{Q}^2(x^1) = \mathbb{E}_{\xi^2(\omega)}\left[Q^2(x^1, \xi^2(\omega))\right]$, and $\mathcal{Q}^\tau(x^{\tau-1}, \xi^{[\tau-1]}(\omega)) = \mathbb{E}_{\xi^\tau(\omega)|\xi^{[\tau-1]}(\omega)}\left[Q^\tau(x^{\tau-1}, \xi^{[\tau]}(\omega))\right]$ for $2 < \tau \leq H$, with

$$Q^{\tau'}(x^{\tau'-1}, \xi^{[\tau']}(\omega)) = \min\Big\{c^{\tau'}(\omega)^\top x^{\tau'}(\omega) + \mathcal{Q}^{\tau'+1}(x^{\tau'}, \xi^{[\tau']}(\omega)) : W^{\tau'}(\omega)x^{\tau'}(\omega) \geq h^{\tau'}(\omega) - T^{\tau'-1}(\omega)x^{\tau'-1}, x^{\tau'}(\omega) \in \mathbb{R}^l_+ \times \mathbb{Z}^{n-l}_+\Big\} \quad \text{for } 2 \leq \tau' \leq H-1,$$

and

$$Q^H(x^{H-1}, \xi^{[H]}(\omega)) = \min\Big\{c^H(\omega)^\top x^H(\omega) : W^H(\omega)x^H(\omega) \geq h^H(\omega) - T^{H-1}(\omega)x^{H-1}, x^H(\omega) \in \mathbb{R}^l_+ \times \mathbb{Z}^{n-l}_+\Big\}.$$

The vectors $c^1 \in \mathbb{R}^n$, $h^1 \in \mathbb{R}^m$, and the matrix $W^1 \in \mathbb{R}^{m\times n}$ are known. For each $\tau = 2, \ldots, H$ and for all ω, $W^\tau(\omega)$ is an $m \times n$ matrix, and $T^{\tau-1}(\omega)$ is an $m \times n$ matrix. $\xi^\tau(\omega)^\top$

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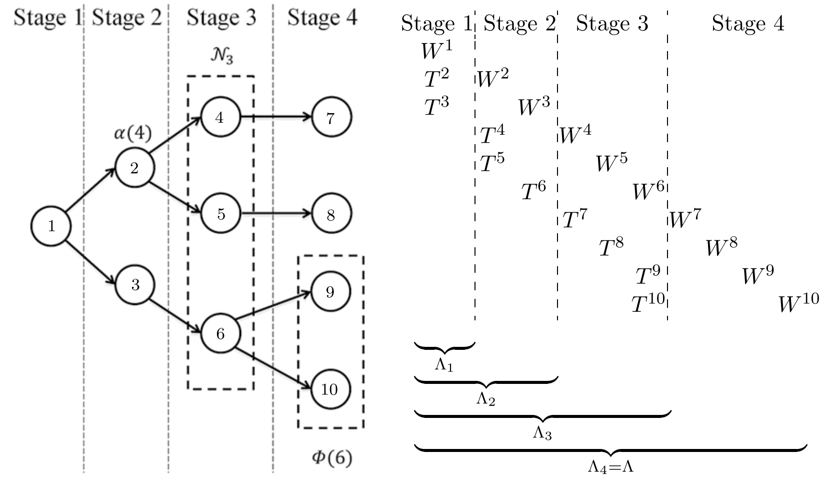

(a) Example of a scenario tree.

(b) Schematic of an extensive-form constraint matrix.

Fig. 1. A scenario tree and a schematic of the corresponding extensive-form constraint matrix.

$= [c^\tau(\omega)^\top, h^\tau(\omega)^\top, T^{\tau-1}_{1,\bullet}(\omega), \ldots, T^{\tau-1}_{m,\bullet}(\omega), W^\tau_{1,\bullet}(\omega), \ldots, W^\tau_{m,\bullet}(\omega)]$ is a random $(n + m + 2mn)$-vector, and $\xi^{[\tau]}(\omega) = (\xi^2(\omega), \ldots, \xi^\tau(\omega))$.

In the following discussion, we assume that $\xi = (\xi^2, \ldots, \xi^H)$ follows a discrete distribution with a finite support Ξ with $|\Xi| = K$. The justification of this assumption was provided by Schultz [14], while a more thorough treatment of multistage stochastic integer programs can be found in Römisch and Schultz [13]. We call $\xi_i = (\xi_i^2, \ldots, \xi_i^H) \in \Xi$ the scenario indexed by $i \in \mathbb{S} = \{1, \ldots, K\}$. Each path from the root node to a leaf node at level H in the scenario tree corresponds to one scenario $i \in \mathbb{S}$. An example of a scenario tree is illustrated in Fig. 1(a).

For a scenario tree $\mathcal{T} = \{\mathcal{N}, \mathcal{A}\}$, let Node 1 be the root node, and $\mathcal{N}_\tau$ be the set of nodes on level $1 \le \tau \le H$, so $\mathcal{N}_1 = \{1\}$. Let $\alpha(k) \in \mathcal{N}$ be the immediate ancestor (or parent) of a non-root node $k \in \mathcal{N} \setminus \{1\}$, $\Phi(k) \subseteq \mathcal{N}$ be the set of immediate children of a node $k \in \mathcal{N}$, and $\rho(k) = \tau$ if $k \in \mathcal{N}_\tau$. Note that $\Phi(k) = \emptyset$ if $\rho(k) = H$. Then the extensive form of (*DEP*) (also called the arborescent form by Dupačová et al. [3]) based on the scenario tree is given by:

$$\min \sum_{k\in\mathcal{N}} p_k c_k^\top x_k \tag{2a}$$

$$\text{s.t.} \quad W^1 x_1 \ge h^1, \tag{2b}$$

$$T^k x_{\alpha(k)} + W^k x_k \ge h^k, \quad \forall\, k \in \mathcal{N} \setminus \mathcal{N}_1, \tag{2c}$$

$$x_k \in \mathbb{R}^l_+ \times \mathbb{Z}^{n-l}_+, \quad \forall\, k \in \mathcal{N}.$$

A schematic of the extensive-form constraint matrix corresponding to the MSP with the scenario tree in Fig. 1(a) is shown in Fig. 1(b). Let Λ denote the extensive-form constraint matrix, and Λ_τ denote the submatrix of Λ up to stage τ as illustrated in Fig. 1(b). Note that $\Lambda = \Lambda_H$. For every $k \in \mathcal{N}$, let A^k denote the submatrix of Λ formed only by W^k and $T^{k'}$ for all $k' \in \Phi(k)$. In particular, we have $A^k = W^k$ if $k \in \mathcal{N}_H$. We are interested in sufficient conditions for the extensive-form constraint matrix of an MSP to be TU. When the right-hand sides are integral, such stochastic programs may be solved as multistage stochastic linear programs, even if there are integrality restrictions.

Definition 1. An $m \times n$ matrix A is *totally unimodular* (TU) if and only if every square submatrix of A has determinant in $\{0, \pm 1\}$.

Theorem 1 (*Hoffman and Kruskal [7]*). *An integral matrix A is totally unimodular if and only if the polyhedron defined by $\{x : Ax \le b, x \ge 0\}$ is integral for all integral b for which it is nonempty, i.e., the extreme points of the polyhedron are integral.*

Theorem 2 (*Ghouila-Houri [6]*). *A $m \times n$ matrix A is TU if and only if for any column subset $J \subseteq \{1, \ldots, n\}$, there exists a partition (J^1, J^2) of J such that*

$$\sum_{j\in J^1} a_{ij} - \sum_{j\in J^2} a_{ij} \in \{0, \pm 1\} \quad \text{for } i = 1, \ldots, m. \tag{3}$$

Definition 2 (*Kong et al. [10]*). Let $\mathcal{A} = \{A_1, \ldots, A_T\}$ be a set of $m \times n$ matrices, and let $\upsilon \in \{0, \pm 1\}^m$. The set $\mathcal{A}$ is *TU with respect to* υ, denoted by $TU(\upsilon)$, if for any column subset $J \subseteq \{1, \ldots, n\}$, there exist partitions (J_t^1, J_t^2), $1 \le t \le T$, such that for $i = 1, \ldots, m$,

$$\sum_{j\in J_t^1} a^t_{ij} - \sum_{j\in J_t^2} a^t_{ij} \in \{0, 1\}, \quad t = 1, \ldots, T, \text{ if } \upsilon_i = -1, \tag{4}$$

$$\sum_{j\in J_t^1} a^t_{ij} - \sum_{j\in J_t^2} a^t_{ij} \in \{0, \pm 1\}, \quad t = 1, \ldots, T, \text{ if } \upsilon_i = 0, \tag{5}$$

and

$$\sum_{j\in J_t^1} a^t_{ij} - \sum_{j\in J_t^2} a^t_{ij} \in \{0, -1\}, \quad t = 1, \ldots, T, \text{ if } \upsilon_i = 1. \tag{6}$$

3. Characterizations of total unimodular multistage stochastic programs

Applying Theorem 2 to (2) yields the following:

Proposition 1. *Let J be a subset of the columns of the extensive-form constraint matrix of an MSP, and for each $k \in \mathcal{N}$, let J_k be the set of the columns in J corresponding to A^k in Λ so that $J = \{J_k\}_{k\in\mathcal{N}}$. Then the MSP is TU if and only if for any J, there exists a partition $(J^1, J^2) := (\{J_k^1\}_{k\in\mathcal{N}}, \{J_k^2\}_{k\in\mathcal{N}})$ such that for $i = 1, \ldots, m$,*

$$\left| \sum_{j\in J_1^1} w^1_{ij} - \sum_{j\in J_1^2} w^1_{ij} \right| \le 1, \tag{7}$$