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The fairest core in cooperative games with transferable utilities

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ABSTRACT

The core and the Shapley value are important solution concepts in cooperative game theory. While the core is designed for the stability of the game, the Shapley value aims for fairness among the players. However, the Shapley value might not lie within the core and a core solution might not be 'fair'. We introduce a new solution concept called the 'fairest core', one that aims for both stability and fairness. We show attractive properties of the fairest core.

1. Introduction and literature review

1.1. Cooperative games and solution concepts

Let n be the number of players and let $\mathcal{N} = \{1, 2, \ldots, n\}$ be the set of all the players. A *coalition* $\mathcal{S}$ is a subset of the players, i.e. $\mathcal{S} \subseteq \mathcal{N}$. The *characteristic function* $v : 2^{\mathcal{N}} \mapsto R$ maps each coalition $\mathcal{S}$ to a real number with $v(\mathcal{S})$ representing the payoff that coalition $\mathcal{S}$ is guaranteed to obtain if all players in $\mathcal{S}$ collaborate, no matter what the remaining players do. A solution of the game $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ is a way to distribute the reward among the players, with x_i being the share for player i. Let us denote $\boldsymbol{x}(\mathcal{S}) = \sum_{i\in\mathcal{S}} x_i$. For each solution $\boldsymbol{x}$, the *excess value* of a coalition $\mathcal{S}$ is defined as $e(\mathcal{S}, \boldsymbol{x}) = v(\mathcal{S}) - \boldsymbol{x}(\mathcal{S})$ which can be viewed as the level of dissatisfaction the players in coalition $\mathcal{S}$ feel over the proposed solution $\boldsymbol{x}$. Solution concepts for cooperative games include:

- An *imputation* is a solution $\boldsymbol{x}$ that satisfies $\boldsymbol{x}(\mathcal{N}) = v(\mathcal{N})$ and $x_i \geq v(\{i\}), \forall i \in \mathcal{N}$.
- The *core* of the game is the set of all imputations $\boldsymbol{x}$ such that $e(\mathcal{S}, \boldsymbol{x}) \leq 0,\ \forall \mathcal{S} \subset \mathcal{N}$. The ϵ-*core* is defined as the set of all imputations $\boldsymbol{x}$ such that $e(\mathcal{S}, \boldsymbol{x}) \leq \epsilon, \forall \mathcal{S} \subset \mathcal{N}$.
- The *least core* is the non-empty ϵ-core with ϵ being the smallest.
- The *Shapley value* is defined as: $\boldsymbol{\phi} = \{\phi_1, \ldots, \phi_n\}$ where

$$\phi_i = \sum_{\mathcal{S}\subset\mathcal{N}} \frac{|\mathcal{S}|!(n - |\mathcal{S}| - 1)!}{n!}(v(\mathcal{S} \cup i) - v(\mathcal{S})),$$

i.e. the Shapley value of player i is the weighted average of the marginal contributions that player i has on all possible coalitions.

1.2. Properties and issues with the core and the Shapley value

Table 1 shows a summary of the properties of three important solution concepts in cooperative games, namely the core, the least core and the Shapley value. The first column includes desirable properties of the payoff distributions.

- **Efficiency**: all three solution concepts are efficient as the entire payoff of the grand coalition is distributed to all the players, i.e. $\boldsymbol{x}(\mathcal{N}) = v(\mathcal{N})$.
- **Existence and uniqueness**: while the Shapley value exists and is unique, the core might not exist in some games. In that case, the least core is introduced as the set of solutions with the least dissatisfaction, i.e. those with the worst excess value being minimised. However, the core, if it exists, and the least core are often non-unique. This might lead to ambiguity, and it is not desirable in real applications because different stakeholders might wish to use different solutions.
- **Stability**: the core, if it exists, is defined to ensure the stability of the game, in the sense that no group of players has the incentive to break out of the grand coalition, because the total share allocated to them is at least the total payoff that they can obtain by forming a coalition themselves. For games with an empty core, the least core is defined in a similar way to minimise the worst dissatisfaction. As the core and the least core are defined to avoid the possibility of groups of players breaking out of the grand coalition, they are referred to as solution concepts with

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Table 1
Properties of the core, the least core and the Shapley value.

	Core	Least core	Shapley value
Efficiency	Yes	Yes	Yes
Existence	Not guaranteed	Yes	Yes
Uniqueness	Not likely	Not likely	Yes
Stability	Yes	Yes	Might not
'Fairness'	Not likely	Not likely	Yes

the stability property. The Shapley value, on the other hand, does not aim for this, and hence it is possible to have unstable Shapley values. We can find many real examples (e.g. in setting ATM interchange fee [2]) and simulated examples (see Section 4) in which the Shapley values are unstable.

- **Fairness w.r.t. dummy players**: the core, the least core, and the Shapley value all suggest that a dummy player who contributes nothing (or a constant value) to all the coalitions receives a share of zero (or the same constant value).
- **Fairness w.r.t. symmetry**: the Shapley value ensures symmetry among the players in the sense that players with the same vectors of marginal contributions should receive the same shares. However, most of the solutions in the core and the least core do not have this property.
- **Fairness w.r.t. monotonicity**: the Shapley value ensures monotonicity among the players, in the sense that a player whose marginal contributions are consistently greater than those of another with respect to any coalition should receive a higher share.
- **Fairness w.r.t. sparsity**: the Shapley value ensures that a player who has any positive contribution to any coalition should receive a positive share, as long as the player does not make any negative contribution to others. Nevertheless, not all solutions in the core and the least core have this property.
- **Additivity**: a nice property of the Shapley value is the additivity property. Given two cooperative games $G_1(\mathcal{N}, v_1)$ and $G_2(\mathcal{N}, v_2)$ that are defined on the same set of players, the Shapley value of the combined game $G_3(\mathcal{N}, v_3 = v_1 + v_2)$ is equal to the sum of the Shapley values of the two games G_1 and G_2. Shapley [9] proves that three axioms on symmetry, additivity and dummy uniquely define the Shapley value. The additivity property is desirable in situations where the characteristic function of the game of interest is a weighted average of several characteristic functions. One such example is the case of a characteristic function that is defined as the expectation over a number of scenarios, each corresponding to a particular value function. In this case, the Shapley value of the stochastic game is simply the average of the Shapley value of the individual scenarios, i.e. $E[\boldsymbol{\phi}[G(\mathcal{N}, \tilde{v})]] = \boldsymbol{\phi}[G(\mathcal{N}, E[\tilde{v}])]$.

2. The fairest core and the fairest least core

We have seen in Section 1.2 that the core, the least core and the Shapley can be grouped into two categories, with the first two solution concepts aiming for stability, while the last one ensures fairness among the players. To apply cooperative game theory successfully to a real application, it would be ideal if the core exists and the Shapley value lies within the core. In this case, choosing the Shapley value would be ideal, since the payoff distribution is unique, fair and stable. It has been shown in [10] that every convex game has a non-empty core and the Shapley value lies within the core. However, it is unfortunate that this is not true for all games. In fact, we will show in an experiment in Section 4 that the Shapley value does not lie within the core in 77% of the random minimum spanning tree games generated. In this case, the question of choosing which solution to balance between fairness and stability is a tricky one. *The goal of this manuscript is to find a solution concept that has desirable properties and at the same time is not too difficult to compute. We introduce the concept of the* ***fairest core****, a* ***unique*** *and* ***stable*** *solution that also has some* ***fairness*** *properties.* This solution is the one that is closest to the Shapley value (to be fair), belongs to the core (to be stable), and is formally defined as follows.

Definition 2.1. Given a game $G(\mathcal{N}, v)$ with non-empty core, the **fairest core** is a solution in the core that has the closest Euclidean distance to the Shapley value, i.e. the fairest core is an optimal solution of the following optimisation problem:

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \|\boldsymbol{x} - \boldsymbol{\phi}\| \\ \text{s.t.} \quad & \boldsymbol{x}(\mathcal{N}) = v(\mathcal{N}), \\ & \boldsymbol{x}(\mathcal{S}) \geq v(\mathcal{S}), \quad \forall \mathcal{S} \subset \mathcal{N}. \end{aligned} \tag{1}$$

In Definition 2.1, the fairest core is defined on a game with non-empty core. For general cases, we extend this to the concept of the fairest least core, which is defined as follows.

Definition 2.2. Given a game $G(\mathcal{N}, v)$, the **fairest least core** is a solution in the least core that has the closest Euclidean distance to the Shapley value, i.e. the fairest least core is an optimal solution of the following optimisation problem:

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \|\boldsymbol{x} - \boldsymbol{\phi}\| \\ \text{s.t.} \quad & \boldsymbol{x}(\mathcal{N}) = v(\mathcal{N}), \\ & \boldsymbol{x}(\mathcal{S}) \geq v(\mathcal{S}) - \epsilon^*, \quad \forall \mathcal{S} \subset \mathcal{N}, \end{aligned} \tag{2}$$

where $\epsilon^* \geq 0$ is the worst excess level for solutions in the least core.

Notice that if the core is non-empty, the two concepts of the fairest core and the fairest least core coincide. In order to compute the fairest least core, we need to compute the least core first. Then we solve the fairest core of a new game whose characteristic function is offset by ϵ^* for all but the grand coalition.

3. Properties of the fairest core and the fairest least core

The first two main results of this research concern the properties of the fairest core and the fairest least core as stated in the following theorems.

Theorem 1. *For games with non-empty core, the fairest core exists and has the following properties:* (a) *it is unique,* (b) *dummy players receive zero shares,* (c) *it is symmetric in the sense that two players with the same vectors of marginal contributions should receive the same shares, and* (d) *it is monotone in the sense that if the marginal contribution of player i is consistently larger than that of player j, i.e.* $v(\mathcal{S} \cup \{i\}) \geq v(\mathcal{S} \cup \{j\}), \forall \mathcal{S} \subset \mathcal{N}$*, then the share of player i should be larger than that of player j.*

Before presenting the proof of Theorem 1, let us explore its implications. First, since the fairest core belongs to the core, it has all the properties of the core, including stability. In addition, it is unique, which means that there is no ambiguity when we refer to this solution concept. It also shares some fairness properties with the Shapley value, namely dummy, symmetry and monotonicity. The monotonicity property implies that the fairest core (and the fairest least core if the core is empty) preserves the ranking between the players if their contributions can be clearly distinguishable. This property is desirable in situations where we want to apply cooperative game theory to find the importance or the ranking among the players, such as in terrorist detection (see [7] for details about the application).

Proof of Theorem 1. a. **Uniqueness**: since the objective function is strictly convex, the optimal solution, if it exists, is unique.

b. **Dummy player gets zero**: since the fairest core belongs to the core, we have $0 = v(\{i\}) \leq x_i = \boldsymbol{x}(\mathcal{N}) - \boldsymbol{x}(\mathcal{N} \setminus i) \leq v(\mathcal{N}) - v(\mathcal{N} \setminus i) = 0$ and hence $x_i = 0$.