Operations Research Letters 43 (2015) 40–45

Contents lists available at ScienceDirect

Operations Research Letters

journal homepage: www.elsevier.com/locate/orl

The truncated normal distribution: Applications to queues with impatient customers

Jamol Pender *

School of Operations Research and Information Engineering, Cornell University, United States

ARTICLE INFO

Article history:
Received 16 April 2014
Received in revised form
30 October 2014
Accepted 30 October 2014
Available online 8 November 2014

Keywords:
Stein's lemma
Truncated normal distribution
Heavy traffic
Abandonment
Queues
Cumulants

ABSTRACT

Motivated by heavy traffic approximations for single server queues with abandonment, we provide an exact expression for the moments of the truncated normal distribution using Stein's lemma. Consequently, our moment expressions provide insight into the steady state skewness and kurtosis dynamics of single server queues with impatient customers. Moreover, based on the truncated normal distribution, we develop a new approximation for single server queues with abandonment in the nonstationary setting. Numerical examples illustrate that our approximation performs quite well.

1. Introduction

The truncated normal distribution is a very important distribution in the world of probability and statistics. It appears quite naturally when the normal distribution itself arises. For example, when one wants to *threshold* or *screen* values from a dataset that is normally distributed, the remaining data has a truncated normal distribution. Therefore to analyze the moments of the remaining data, one needs to study the moments of the truncated normal distribution.

Most if not all of the available literature tends to focus on the mean and variance of the truncated normal distribution. This is partially motivated from the statistical community since they are interested in obtaining unbiased mean and variance estimators for data that is screened or thresholded. See for example [1–3,5]. In this paper, we not only provide exact expressions for the skewness and kurtosis, but also provide any moment of the truncated normal distribution. Later in the paper, we also use the truncated normal distribution to approximate the nonstationary single server queue with abandonment.

Although there is substantial motivation to study the moments of the truncated normal distribution from a statistical perspective, we are primarily motivated by developing approximations for the cumulant moments of queues with impatient customers. There is a large and growing literature on queues with impatient customers, for instance, [21,22] show that the truncated normal distribution arises as the heavy traffic diffusion limit for the stationary single server queue with impatient customers. More recently, [6] showed that the truncated normal distribution is the heavy traffic limit of ticket queues where customers are unobservable. In [22] they consider a GI/GI/1 + GI queueing model with abandonment. They assume that the server works at rate one under the FIFO discipline. The primitives of the model include three independent sequences of non-negative i.i.d. random variables for the inter arrival times, service times, and abandonment times. We assume that the service times have mean $\frac{1}{\mu}$ and coefficient of variation σ_s. The inter arrival times have mean $\frac{1}{\lambda} = \frac{1}{\mu+\beta\cdot\sqrt{\mu}}$ and coefficient of variation σ_a where β is the heavy traffic parameter. Lastly, we assume that the abandonment can have any distribution where the derivative of cdf evaluated at zero is strictly positive with value θ. The main theorem proved in [22] says the following:

Theorem 1.1 *([22]). If*

$$\tilde{Q}^n(0) \Rightarrow \tilde{Q}_0, \quad \text{as } n \to \infty,$$

then we have the following convergence for the queue length process and generalized linear regulator mapping $(\tilde{Q}^n, \tilde{Y}^n)$ as described in Eq. 3.3 *in* [22]

$$(\tilde{Q}^n, \tilde{Y}^n) \Rightarrow (\tilde{Q}, \tilde{Y}), \quad \text{as } n \to \infty,$$

* Correspondence to: 206 Rhodes Hall, Ithaca, NY 14850, USA.
E-mail address: jjp274@cornell.edu.

http://dx.doi.org/10.1016/j.orl.2014.10.008
0167-6377/

where $\tilde{Q}(0)$ is equal in distribution to $\tilde{Q}_0$ and together $\tilde{Q}$ and $\tilde{Y}$ obey the following stochastic differential equation:

$$d\tilde{Q}(t) = (-\beta + \theta \cdot \tilde{Q}(t))dt + \sigma \cdot dB(t) + d\tilde{Y}(t)$$

where β is the scaled heavy traffic scaling parameter, θ is the derivative of the abandonment distribution at zero, and $\sigma^2 = \sigma_a^2 + \sigma_s^2$ is the sum of the arrival and service distributions coefficient of variation.

The process $\tilde{Q}(t)$ is known as a regulated Ornstein–Uhlenbeck (ROU) process and the steady state distribution of $\tilde{Q}(t)$ is a truncated normal random variable that is conditioned or regulated to be in the interval $(0, \infty)$

$$\tilde{Q}(\infty) = \text{Normal}\left(\frac{\beta}{\theta}, \frac{\sigma^2}{2\theta}, 0, \infty\right). \tag{1.1}$$

In the work of [6,22], they only analyze the steady state mean dynamics. However, it is important to analyze higher cumulants such as the variance, skewness, and kurtosis as they provide essential insights into the behavior of the queueing process. Since the Gaussian is defined to have zero skewness and zero excess kurtosis, when the skewness and kurtosis are far from zero, it implies that a Gaussian approximation of the dynamics might not be appropriate. In fact, the work of [9–11,15–19] shows that the skewness and kurtosis can play a significant role in estimating queueing performance. Thus, we believe that our exact expressions for the higher cumulants of the truncated normal will give us insight into the dynamics of queues with impatient customers.

2. Stein's lemma and main results

2.1. Stein's lemma

In this section, we give a brief overview of Hermite polynomials and Stein's lemma [20], which are important ingredients for deriving our exact expressions for the moments of the truncated normal distribution. The probabilistic Hermite polynomials as described in [14] are defined as:

$$h_n(x) = \frac{1}{\varphi(x)} \cdot \left(-\frac{d}{dx}\right)^n \varphi(x).$$

The first four Hermite polynomials are

$$h_0(x) = 1, \qquad h_1(x) = x, \qquad h_2(x) = x^2 - 1,$$
$$h_3(x) = x^3 - 3x,$$

and in general they solve the recurrence relation

$$h_{n+1}(x) = x \cdot h_n(x) - n \cdot h_{n-1}(x).$$

We have the following Hermite polynomial generalization of Stein's lemma; however, for the remainder of the paper, the random variable X is a standard Gaussian random variable.

Lemma 2.1. *If X is a standard Gaussian random variable and $E[f^{(n)}(X)] < \infty$, then*

$$E[f(X) \cdot h_n(X)] = E[f^{(n)}(X)]$$

where f is any generalized function and $f^{(n)}$ is the nth derivative of the function f.

For example, since $\{X \geq \chi\}$ is a generalized function, Stein's lemma can be used to obtain

$$E[X \cdot \{X \geq \chi\}] = E\left[\delta_\chi(X)\right] = \varphi(\chi),$$

or for $n \geq 1$

$$E[h_n(X) \cdot \{X \geq \chi\}] = E\left[h_{n-1}(X) \cdot \delta_\chi(X)\right] = h_{n-1}(\chi) \cdot \varphi(\chi),$$

where we define φ and Φ to be the density and the cumulative distribution functions, respectively, for $X \sim \text{Normal}(0, 1)$, i.e.,

$$\varphi(x) \equiv \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \qquad \Phi(x) \equiv \int_{-\infty}^{x} \varphi(y)\, dy,$$

and let $\overline{\Phi}(x) \equiv \int_x^\infty \varphi(y)\, dy$.

In addition to the derivative properties of the Hermite polynomials, it is well known from [4] that the probabilistic Hermite polynomials have the following explicit form in terms of standard polynomials,

$$h_n(x) = n! \cdot \sum_{m=0}^{\lfloor n/2 \rfloor} \frac{(-1)^m}{m! \cdot (n-2m)!} \cdot \frac{x^{n-2m}}{2^m}. \tag{2.2}$$

However, the above relation of Eq. (2.2) can be inverted to represent any polynomial in terms of the Hermite polynomials as the next theorem shows.

Theorem 2.2. *Any polynomial has the following decomposition representation in terms of the probabilist Hermite polynomials*

$$X^n = n! \cdot \sum_{m=0}^{\lfloor n/2 \rfloor} \frac{2^{-m}}{m! \cdot (n-2m)!} \cdot h_{n-2m}(X).$$

Proof. This proof follows from induction and exploiting the Rodrigues recursion relation for Hermite polynomials. We also provide a proof of this result in an online appendix to this paper found on the author's website. □

This representation of any polynomial in terms of a sum of Hermite polynomials will be useful since any moment of a Gaussian random variable can be computed by the last coefficient of the sum since all Hermite polynomials where $n \geq 1$ have expectation zero. Now with our review of Hermite polynomials and their properties, we give our main result.

Theorem 2.3. *Suppose that Q has a normal distribution with mean q and variance v; then the nth conditional moment has representation*

$$E[Q^n | a \leq Q \leq b] = \frac{\sum_{j=0}^{n} a_j \cdot \left(\sum_{m=0}^{\lfloor j/2 \rfloor} b_{jm} \cdot (h_{n-2m-1}(\chi) \cdot \varphi(\chi) - h_{n-2m-1}(\psi) \cdot \varphi(\psi)) \right)}{\Phi(\psi) - \Phi(\chi)}$$

where we define

$$a_j = \binom{n}{j} \cdot \sqrt{v^j} \cdot q^{n-j} \cdot j! \quad \text{and} \quad b_{jm} = \frac{2^{-m}}{m! \cdot (j-2m)!},$$
$$(h_{-1}(\chi) \cdot \varphi(\chi) - h_{-1}(\psi) \cdot \varphi(\psi)) = \Phi(\psi) - \Phi(\chi),$$

and

$$\chi = \frac{a-q}{\sqrt{v}} \quad \text{and} \quad \psi = \frac{b-q}{\sqrt{v}}.$$

Proof. We prove this result in an online appendix to this paper found on the author's website. □

As a result of the above expression, we have the following corollary, which gives explicit expressions for the mean, variance, skewness, and kurtosis of the truncated normal distribution.

Corollary 2.4. *Eqs. (2.3), (2.4),* Skew$[Q | a \leq Q \leq b]$ *and* Kurt$[Q | a \leq Q \leq b]$ *are given in Box I.*

Proof. After some tedious calculations which we omit for brevity, the proof follows from using Theorem 2.3 and understanding the definitions of the variance, skewness, and kurtosis of random variables. □