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A short convex-hull proof for the all-different system with the inclusion property

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ABSTRACT

An all-different constraint on some discrete variables imposes the condition that no two variables take the same value. A linear-inequality description of the convex hull of solutions to a system of all-different constraints is known under the so-called *inclusion property*: the convex hull is the intersection of the convex hulls of each of the all-different constraints of the system. We give a short proof of this result, which in addition shows the total dual integrality of the linear system.
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1. Introduction

In many combinatorial optimization problems one needs to impose one or more *all-different constraints*, i.e., conditions of the following type: for a given finite (sub)family of discrete variables, no two variables can be assigned the same value. All-different constraints arise, for instance, in problems related to timetabling, scheduling, manufacturing, and in several variants of the assignment problem (see, e.g., [6,9] and the references therein).

Though all-different constraints are mainly studied in the context of Constraint Programming (see, e.g., [8]), when dealing with a problem that can be modeled as an integer linear program it is useful to have information on the polyhedral structure of the feasible solutions to a system of all-different constraints. For this reason, several authors studied linear-inequality formulations for the convex hull of solutions to a single all-different constraint or a system of all-different constraints [3,5,6,9]. We remark that in some cases these descriptions are extended formulations, i.e., they make use of additional variables; however, here we are only interested in the description of the convex hull in the original space of variables.

If n variables $x_1, \dots, x_n$ can take values in a finite domain $D \subseteq \mathbb{R}$ and an all-different constraint is imposed on them, we will write (following the notation in [6])

$$\{x_1, \dots, x_n\}_{\neq} \tag{1}$$

$$x_1, \dots, x_n \in D. \tag{2}$$

Williams and Yan [9] proved that if $D = \{1, \dots, d\}$ for some positive integer d, then the convex hull of the vectors that satisfy (1)–(2) is described by the linear system

$$\sum_{j \in S} x_j \geq f(S), \quad S \subseteq [n], \tag{3}$$

$$\sum_{j \in S} x_j \leq g(S), \quad S \subseteq [n], \tag{4}$$

where $[n] = \{1, \dots, n\}$ and, for $S \subseteq [n]$,

$$f(S) = \frac{|S|(|S|+1)}{2}, \qquad g(S) = |S|(d+1) - f(S). \tag{5}$$

Note that $f(S)$ is the sum of the $|S|$ smallest positive integers, while $g(S)$ is the sum of the $|S|$ largest integers that do not exceed d, therefore inequalities (3)–(4) are certainly valid for every vector x satisfying (1)–(2). This result extends to an arbitrary finite domain $D \subseteq \mathbb{R}$ (with $|D| \geq n$) by defining $f(S)$ (resp., $g(S)$) as the sum of the $|S|$ smallest (resp., largest) elements in D, for every $S \subseteq [n]$. Note however that in the following we assume $D = \{1, \dots, d\}$ for some positive integer d, while we will consider the case of a generic finite domain D in a final remark.

Williams and Yan [9] showed that if $d > n$ then all inequalities (3)–(4) are facet-defining, thus the convex hull of (1)–(2) needs an

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exponential number of inequalities to be described in the original space of variables $x_1, \dots, x_n$. However, they also gave polynomial-size extended formulations for the convex hull of (1)–(2).

When $d = n$, (1)–(2) is the set of permutations of the elements in $[n]$, and its convex hull is called permutahedron. In this case, the whole family of inequalities (4) can be dropped and replaced by the equation $\sum_{j\in[n]} x_j = f([n])$. The permutahedron admits an extended formulation with $O(n \log n)$ constraints and variables [2].

System (3)–(4) not only defines an integral polyhedron, but it also has the stronger property of being *totally dual integral*. We recall that a linear system of inequalities $Ax \le b$ is said to be totally dual integral if for every integer vector c such that the linear program $\max\{cx : Ax \le b\}$ has finite optimum, the dual linear program has an optimal solution with integer components. It is known that if $Ax \le b$ is totally dual integral and b is an integer vector, then the polyhedron defined by $Ax \le b$ is integral (see, e.g., [7, Theorem 5.22]). The total dual integrality of system (3)–(4) follows immediately from the fact that f (resp., g) is a supermodular (resp., submodular) function, along with a classical result on polymatroid intersection [1] (see also [7, Theorem 46.2]). An explicit proof of the total dual integrality of (3)–(4) is given in [5].

In a more general setting, we might have $m \ge 1$ all-different constraints, each enforced on a different subset of variables $N_i \subseteq [n]$, $i \in [m]$. In this case, we have the system of conditions

$$\{x_j : j \in N_i\}_{\neq}, \quad i \in [m], \tag{6}$$

$$x_1, \dots, x_n \in D. \tag{7}$$

The following inequalities are of course valid for the convex hull of solutions to (6)–(7):

$$\sum_{j\in S} x_j \ge f(S), \quad S \subseteq N_i,\ i \in [m], \tag{8}$$

$$\sum_{j\in S} x_j \le g(S), \quad S \subseteq N_i,\ i \in [m]. \tag{9}$$

However, the above constraints do not give, in general, the convex hull of the vectors that satisfy (6)–(7). Furthermore, there are even examples in which some integer solutions to system (8)–(9) do not lie in the convex hull of the points satisfying (6)–(7). Therefore it is natural to ask which conditions ensure that the above system provides the convex hull of the vectors satisfying (6)–(7).

A special case, studied in [6], in which constraints (8)–(9) do yield the convex hull of solutions to (6)–(7) is now described. Define $N = [n]$ and assume that $N = T \cup U$, where T and U are disjoint nonempty subsets of N. Define $T_i = N_i \cap T$ and $U_i = N_i \cap U$ for $i \in [m]$. If the T_i's form a monotone family of subsets ($T_1 \supseteq T_2 \supseteq \cdots \supseteq T_m$) and the U_i's are pairwise disjoint, then Magos et al. [6] say that the *inclusion property* holds. They showed that in this case inequalities (8)–(9) provide the convex hull of solutions to (6)–(7). We remark that (to the best of the author's knowledge) this is the only nontrivial case in which formulation (8)–(9) is known to define the convex hull of the all-different system.

The proof of Magos et al. [6] is rather lengthy and involved (overall, it consists of about 25 pages). The purpose of this note is to give a shorter proof of their result. Indeed, we show something more: we prove that, under the inclusion property, system (8)–(9) is totally dual integral. Our proof is an extension of the classical approach to show the total dual integrality of polymatroids (see, e.g., [7, Chapter 44]). Specifically, in Section 2 we describe a greedy algorithm that solves linear optimization over (8)–(9), under the inclusion property. The correctness of the algorithm is shown in Section 3 by completing the feasible solution returned by the algorithm with a dual solution such that the complementary slackness conditions are satisfied. The result of Section 3 also implies the total dual integrality of system (8)–(9), as the dual solution is integer whenever the primal objective function coefficients are all integers. We conclude in Section 4 with an extension of the result, which in particular can be used to deal with a generic finite domain D.

2. Primal algorithm

Assume that the inclusion property holds for an all-different system (6)–(7). Recall that:

- $N = [n] = T \cup U$, with T and U disjoint and nonempty;
- $T_i = N_i \cap T$ and $U_i = N_i \cap U$ for $i \in [m]$;
- $T_1 \supseteq \cdots \supseteq T_m$;
- $U_i \cap U_j = \varnothing$ for all distinct $i, j \in [m]$.

Wlog, $N = N_1 \cup \cdots \cup N_m$ and $T = T_1 = [t]$ for some positive integer t. Also, recall that $D = [d]$. We assume that $d \ge \max_{i\in[m]} |N_i|$, otherwise both (6)–(7) and (8)–(9) are infeasible. We use the notation $t_i = |T_i|$ and $u_i = |U_i|$ for $i \in [m]$. Furthermore, we will sometimes identify an index $j \in N$ with the corresponding variable x_j; e.g., we will indifferently say "the indices in T" or "the variables in T".

Consider the problem of minimizing a linear objective function cx over the polytope defined by (8)–(9), where c is a row-vector in $\mathbb{R}^n$. If we define $\mathcal{S} = \bigcup_{i\in[m]}\{S : S \subseteq N_i\}$, the problem can be written as follows:

$$\min \quad cx \tag{10}$$

$$\text{s.t.} \quad \sum_{j\in S} x_j \ge f(S), \quad S \in \mathcal{S}, \tag{11}$$

$$-\sum_{j\in S} x_j \ge -g(S), \quad S \in \mathcal{S}. \tag{12}$$

We give a greedy algorithm that solves the above linear program for an arbitrary $c \in \mathbb{R}^n$. Since the polyhedron defined by (11)–(12) contains all vectors satisfying (6)–(7), and since we will show that the solution returned by the algorithm satisfies (6)–(7), this will prove that system (11)–(12) (i.e., system (8)–(9)) defines the convex hull of (6)–(7). The algorithm that we present can be seen as an extension of the greedy algorithm for polymatroids (see, e.g., [7, Chapter 44]), and also as an extension of the algorithm given in [5] for the case $m = 1$.

The procedure is shown in Algorithm 1 and is now illustrated. Throughout the algorithm, we maintain d *clusters* of variables $V_1, \dots, V_d$, i.e., d (possibly empty) disjoint subsets of N gathering those variables that will be assigned the same value at the end of the algorithm. At the beginning (lines 2–3) we have t nonempty clusters $V_1, \dots, V_t$, where $V_j = \{j\}$ for $j \in [t]$, while the other clusters $V_{t+1}, \dots, V_d$ are empty. Thus every variable in $T = [t]$ is assigned to a different cluster (as these variables are not allowed to take the same value because of the first all-different constraint), while the variables in U are not assigned to any cluster. During the execution of the algorithm, each variable in U will be assigned to a cluster, and no variable will be ever moved from one cluster to another.

Notation $r(j)$ indicates the index of the cluster to which variable x_j is assigned. With each cluster V_j, $j \in [d]$, we associate a *pseudo-cost* γ_j, which is the sum of the costs of all variables in the cluster. The pseudo-cost of an empty cluster is zero.

For $i = 1, \dots, m$, at the ith iteration of the algorithm we assign each variable in U_i to a different cluster (lines 4–11), as we now explain. Because of the ith all-different constraint, a variable in U_i cannot be assigned to a cluster containing a variable in T_i. Note that for $j \in T_i$, the cluster containing j is V_j. Therefore only the clusters V_j with $j \in [d] \setminus T_i$ are *feasible* for the variables in U_i. Lines 5–6 order

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