



## Vote trading and subset sums



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### ARTICLE INFO

#### Article history:

Received 17 September 2014

Received in revised form

6 November 2014

Accepted 8 December 2014

Available online 17 December 2014

#### Keywords:

Vote swapping

Vote trading

Manipulation

Subset sum problem

### ABSTRACT

We analyze the complexity of vote trading problems with equal-sized voting districts. For two allied vote-swapping parties, the problem is polynomially solvable. For three parties, the problem is NP-complete.

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### 1. Introduction

In the 2000 presidential elections in the USA, the essential decision was to be made between the Republican candidate George W. Bush (the incumbent governor of Texas and son of former president George Bush) and the Democratic candidate Al Gore (the incumbent Vice President). There also was a third candidate, the Green Party nominee Ralph Nader, but he was a clear outsider and in the end received less than 3% of the total votes. (We remind the reader that in US presidential elections the country is divided up into states each with a certain number of electoral votes. The number of electoral votes in a state is roughly proportional to a state's population. The candidate who wins most of a state's popular votes wins all of this state's electoral votes. The candidate who wins a majority of the electoral votes wins the election.)

Now the 2000 presidential elections were expected to be very close between Gore and Bush, as they both attracted similar numbers of voters in the polls leading up to the election. Since Gore and Nader had similar political positions, the supporters of Gore were concerned that Nader was potentially taking votes away from Gore in swing states and that Gore could thus lose the election because of the presence of Nader. Therefore *vote trading* was introduced, an Internet mechanism with the idea that Gore should

become stronger in states where this would help him, while Nader should become stronger in states where this would not hurt Gore. The mechanism paired up Gore supporters in states that strongly favored Bush with Nader supporters in states where Gore and Bush were close. Each pair would agree to swap votes: the Gore supporter would vote for Nader and the Nader supporter would vote for Gore. In this way Gore would have a better chance to win in closely contested states, while Nader would still get the same number of votes nationally, which was important to him for future funding. As this mechanism preserves the total number of votes received by both candidates across states, it was attractive to electors and has also been implemented since in subsequent elections.

This vote trading mechanism raises a number of ethical issues that are discussed for instance by Randazza [8], Hartvigsen [6] and Bervoets and Merlin [2]. Bervoets and Merlin [1] perform an axiomatic analysis of the problem of vote trading, as well as of the closely related problem of gerrymandering. On the algorithmic side, vote trading triggers the investigation of optimal strategies for the involved candidates. Hartvigsen [5] presents a mathematical model for vote trading problems and analyzes a variety of algorithmic and combinatorial concepts in this area. In particular, Hartvigsen establishes the NP-hardness of optimal vote trading in the case where two allied parties *B* and *C* are swapping votes with the goal of weakening a third party *A* and where different voting districts may have different sizes. (This last assumption on the district sizes is open to criticism, as voting districts are usually designed to be of equal size.)

*Our contribution.* In this short technical note, we discuss vote trading in the cases where all the voting districts are of identical

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<http://dx.doi.org/10.1016/j.orl.2014.12.007>

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size. We show that then the best vote trading can be found in polynomial time, if there are only two allied parties that are swapping votes; note that this result is in contrast to the result with arbitrarily sized districts of Hartvigsen [5]. For three allied parties, however, also the problem with identical sized districts becomes NP-complete. Our results draw a sharp separation line between easy and hard cases. Furthermore, they yield yet another example for Lawler's mystical power of twoness; see Lenstra [7].

The note is organized as follows. Section 2 discusses a variant of the classical subset sum problem, and identifies a polynomially solvable special case of this variant. Section 3 establishes a connection between vote swapping with two allied parties (and equal-sized voting districts) and the subset sum variant from Section 2; this connection yields the polynomial time result. Section 4 establishes NP-hardness of vote swapping with three allied parties (and equal-sized voting districts).

## 2. A subset sum variant

Subset sum problems are centered around  $n$  items with positive integer sizes  $u_1, \dots, u_n$ , and ask certain questions about the values attained by  $u(I) := \sum_{i \in I} u_i$  as  $I$  ranges over the item subsets  $I \subseteq \{1, \dots, n\}$ . As a rule of thumb, subset sum problems are computationally intractable. For example, the problem of deciding whether  $u(I)$  attains all integer values between two given bounds  $V^-$  and  $V^+$  is  $\Pi_2^P$ -complete; see Eggermont and Woeginger [3]. As another example, the problem of deciding whether  $u(I)$  attains some concrete given integer goal value  $V$  is NP-complete; see Garey and Johnson [4]. This latter example with goal value  $V$  constitutes the classical SUBSET-SUM problem, which plays a fundamental and prominent role in the area of combinatorial optimization.

In general, we should not expect to find simple certificates for NO-instances of SUBSET-SUM that are easy to verify (as this would imply  $\text{NP} = \text{coNP}$ ). But for certain well-behaved special cases the NO-instances are easy to recognize. For example, if all the item sizes  $u_1, \dots, u_n$  are even while the goal value  $V$  is odd, then the answer certainly must be NO. For another example, if the sum of the largest three values among  $u_1, \dots, u_n$  is strictly smaller than  $V$  while the sum of the smallest four values among  $u_1, \dots, u_n$  is strictly larger than  $V$ , then the answer also must be NO. In the rest of this section, we will consider the following subset sum variant and we will identify a polynomially solvable special case that is centered around this latter observation.

Problem: SUBSET-SUM INTERVAL

Instance: Items with positive integer sizes  $u_1, \dots, u_n$ ; two integers  $V^- \leq V^+$ .

Question: Does there exist  $I \subseteq \{1, \dots, n\}$  with  $V^- \leq u(I) \leq V^+$ ?

Note that for  $V^- = V^+$ , problem SUBSET-SUM INTERVAL boils down to problem SUBSET-SUM; consequently SUBSET-SUM INTERVAL is NP-complete.

**Lemma 2.1.** *The special case of SUBSET-SUM INTERVAL with*

$$V^+ - V^- \geq \max_{i=1}^n u_i - \min_{i=1}^n u_i \quad (1)$$

*is polynomially solvable.*

**Proof.** First renumber the items so that  $u_1 \leq u_2 \leq \dots \leq u_n$  holds. For  $1 \leq p \leq n$  define  $S_p^{\min} = \sum_{i=1}^p u_i$  and  $S_p^{\max} = \sum_{i=n-p+1}^n u_i$  as the sum of the  $p$  smallest respectively the  $p$  largest items; furthermore let  $S_0^{\max} = -\infty$  and  $S_{n+1}^{\min} = +\infty$ . Consider the following polynomial time algorithm:

- Determine the largest index  $r$  ( $0 \leq r \leq n$ ) for which  $S_r^{\max} < V^-$ .
- If  $V^+ < S_{r+1}^{\min}$  then output NO, and otherwise output YES.

First assume that the algorithm outputs NO, so that  $S_r^{\max} < V^- \leq V^+ < S_{r+1}^{\min}$  holds. Note that any set  $I \subseteq \{1, \dots, n\}$  with cardinality  $|I| \leq r$  satisfies  $u(I) \leq S_r^{\max} < V^-$ , and that any set  $I$  with cardinality  $|I| > r$  satisfies  $u(I) \geq S_{r+1}^{\min} > V^+$ . Hence there is no  $I$  with  $V^- \leq u(I) \leq V^+$ , and the output of the algorithm is correct.

Next assume that the algorithm outputs YES. This implies  $r + 1 \leq n$ , and also  $V^- \leq S_{r+1}^{\max}$  and  $S_{r+1}^{\min} \leq V^+$ . If  $S_{r+1}^{\min} \geq V^-$  then the set  $I = \{1, \dots, r + 1\}$  constitutes a feasible solution, and if  $S_{r+1}^{\max} \leq V^+$  then the set  $I = \{n - r + 1, \dots, n\}$  constitutes a feasible solution. It remains to consider the cases with

$$S_{r+1}^{\min} < V^- \leq V^+ < S_{r+1}^{\max}.$$

We start with the set  $I = \{1, \dots, r + 1\}$  that contains the  $r + 1$  smallest items, and then step by step replace some item by a larger one. Every step raises  $u(I)$  by at most  $\max_i u_i - \min_i u_i$ , so that by (1) the value  $u(I)$  eventually must fall between the bounds  $V^-$  and  $V^+$ . Hence also in this case, the output of the algorithm is correct.  $\square$

## 3. The vote trading problem

In this section, we discuss the following special case of vote trading with three political parties  $A, B, C$  and with  $m$  equal-sized voting districts. The number of voters in the  $i$ th district that respectively vote for  $A, B, C$  is denoted by  $a_i, b_i, c_i$ . As all voting districts have equal size  $s$ , we have  $a_i + b_i + c_i = s$  for  $1 \leq i \leq m$ . Every district is won by the party that receives the relative majority of votes. For the sake of simplicity we assume that ties are always broken to the disadvantage of party  $A$ ; therefore party  $A$  wins the  $i$ th district if and only if  $a_i > \max\{b_i, c_i\}$  holds.

The question is whether parties  $B$  and  $C$  can repartition their votes such that they reach the relative majority in at least  $k$  of the districts. This reflects the specific situation of the US presidential elections (involving Bush, Gore, and Nader) as discussed at the beginning of this paper: one might be pretty sure that even after vote trading between  $B$  and  $C$  (Gore and Nader) only one of  $B$  and  $C$  (Gore) will win any district. Under this assumption, keeping  $A$  below 50% will do the job. Here is a formal description of this question.

Problem: VOTE TRADING

Instance: Non-negative integers  $a_1, \dots, a_m, b_1, \dots, b_m$ , and  $c_1, \dots, c_m$  with  $a_i + b_i + c_i = s$  for  $1 \leq i \leq m$ ; an integer  $k$ .

Question: Do there exist non-negative integers  $b'_1, \dots, b'_m$  and  $c'_1, \dots, c'_m$ , with  $\sum_{i=1}^m b'_i = \sum_{i=1}^m b_i$ , and  $\sum_{i=1}^m c'_i = \sum_{i=1}^m c_i$ , and  $b'_i + c'_i = b_i + c_i$  for  $1 \leq i \leq m$ , such that the following holds: there exists an index set  $I \subseteq \{1, \dots, m\}$  with  $|I| = k$ , such that  $a_i \leq \max\{b'_i, c'_i\}$  for all  $i \in I$ ?

For later reference, we note that  $a_i + b_i + c_i = s$  for  $1 \leq i \leq m$  implies

$$\sum_{i=1}^m a_i + \sum_{i=1}^m b_i + \sum_{i=1}^m c_i = ms. \quad (2)$$

Furthermore, we will assume without loss of generality that the numbering of the districts satisfies

$$a_1 \leq a_2 \leq \dots \leq a_{m-1} \leq a_m. \quad (3)$$

Under (3) it is straightforward to see that parties  $B$  and  $C$  can win  $k$  districts if and only if they can win the first  $k$  districts. Finally, we

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