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Some results for quadratic problems with one or two quadratic constraints

Marco Locatelli

Dipartimento di Ingegneria dell'Informazione, Università di Parma Via G.P. Usberti, 181/A, 43124 Parma, Italy

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ABSTRACT

In this paper we discuss problems with quadratic objective function, one or two quadratic constraints, and, possibly, some additional linear constraints. In particular, we consider cases where the Hessian of the quadratic functions are simultaneously diagonalizable, so that the objective and constraint functions can all be converted into separable functions. We give conditions under which a simple convex relaxation of these problems returns their optimal values.

1. Introduction

In this paper we discuss simple convex relaxations for some quadratic programming problems. We will first consider problems with a quadratic objective function and two quadratic constraints

$$\begin{aligned} \min \quad & \frac{1}{2}\mathbf{x}^T\mathbf{Q}_0\mathbf{x} + \mathbf{q}_0^T\mathbf{x} \\ & \frac{1}{2}\mathbf{x}^T\mathbf{Q}_1\mathbf{x} + \mathbf{q}_1^T\mathbf{x} \le u \\ & \frac{1}{2}\mathbf{x}^T\mathbf{Q}_2\mathbf{x} + \mathbf{q}_2^T\mathbf{x} \le v. \end{aligned} \tag{1}$$

Later on we will also discuss cases with a single quadratic constraint but additional linear constraints. Problem (1) is, in general, a difficult one, although some subclasses are polynomially solvable. For instance, in [14] it is shown that a semidefinite relaxation allows to solve it when $\mathbf{q_i} = \mathbf{0}$, $i = 0, 1, 2$, i.e., when we have no linear term. In [2] the case of two-sided indefinite quadratic constraints is shown to be solvable in polynomial time under a suitable assumption. This case is also discussed in [10]. Approximation results for the case with an arbitrary number of quadratic equality constraints with a diagonal Hessian and no linear terms, and with additional bound constraints, are discussed in [13]. Theorems of the alternative for inequality systems with two or three quadratic inequalities are discussed in [8].

In the recent paper [1], the case where the three matrices $\mathbf{Q}_i$, $i = 0, 1, 2$, are simultaneously diagonalizable (SD in what follows) is considered. This means that there exists a nonsingular matrix $\mathbf{S}$ such that

$$\mathbf{S}^T\mathbf{Q}_i\mathbf{S} = \mathbf{D}^i, \quad i = 0, 1, 2,$$

where each matrix $\mathbf{D}^i$, $i = 0, 1, 2$, is diagonal. Cases of SD matrices, taken from [1], include:

$$\begin{aligned} & \mathbf{Q}_j = \mathbf{O} \quad \text{for some } j \in \{0, 1, 2\}, \\ & \mathbf{Q}_k, \mathbf{Q}_h, \quad h, k \in \{0, 1, 2\}, h, k \ne j, \ h \ne k, \ \text{are SD} \end{aligned} \tag{2}$$

$$\begin{aligned} & \mathbf{Q}_j = \mathbf{I} \quad \text{for some } j \in \{0, 1, 2\}, \\ & \mathbf{Q}_k, \mathbf{Q}_h, \ h, k \in \{0, 1, 2\}, h, k \ne j, \ h \ne k, \ \text{commute} \end{aligned} \tag{3}$$

$$\begin{aligned} & \mathbf{Q}_h = -\mathbf{Q}_k, \quad h, k \in \{0, 1, 2\}, \ h \ne k, \\ & \mathbf{Q}_j, \mathbf{Q}_h, j \ne h, k, \ \text{are SD.} \end{aligned} \tag{4}$$

Notice that here and in what follows $\mathbf{O}$ denotes the null matrix. Under the assumption that the three matrices $\mathbf{Q}_i$, $i = 0, 1, 2$, are SD, problem (1) can be rewritten as follows, where the objective and constraint functions are all separable

$$\begin{aligned} \min \quad & \frac{1}{2}\sum_{i=1}^n q_i y_i^2 + \sum_{i=1}^n c_i y_i \\ & \frac{1}{2}\sum_{i=1}^n \eta_i y_i^2 + \sum_{i=1}^n d_i y_i \le u \\ & \frac{1}{2}\sum_{i=1}^n \gamma_i y_i^2 + \sum_{i=1}^n b_i y_i \le v. \end{aligned} \tag{5}$$

E-mail address: locatell@ce.unipr.it.

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In what follows we will assume that $d_i = 0,\ i = 1, \dots, n$. Note that this can always be made true if $\eta_i \neq 0$. Indeed, in this case we have

$$\eta_i y_i^2 + d_i y_i = \eta_i y_i^2 + d_i y_i + \frac{d_i^2}{4\eta_i} - \frac{d_i^2}{4\eta_i} = \frac{1}{2\eta_i}(2\eta_i y_i + d_i)^2 - \frac{d_i^2}{4\eta_i},$$

so that, after the change of variable $y_i' = 2\eta_i y_i + d_i$, we have no linear term involving y_i' in the constraint, i.e., we can set $d_i = 0$. We do not discuss in what follows the case when $\eta_i = 0,\ d_i \neq 0$, since it can be dealt with in a completely analogous way.

Now, after introducing the variables $z_i,\ i = 1, \dots, n$, problem (5), with $d_i = 0,\ i = 1, \dots, n$, can be rewritten as

$$\begin{aligned} \min\quad & \sum_{i=1}^n q_i z_i + \sum_{i=1}^n c_i y_i \\ & \sum_{i=1}^n \eta_i z_i \leq u \\ & \sum_{i=1}^n \gamma_i z_i + \sum_{i=1}^n b_i y_i \leq v \\ & \frac{1}{2} y_i^2 = z_i \qquad i = 1, \dots, n. \end{aligned} \tag{6}$$

If we replace $=$ with $\leq$ in the last constraints, we are led to the following relaxation of (5):

$$\begin{aligned} \min\quad & \sum_{i=1}^n q_i z_i + \sum_{i=1}^n c_i y_i \\ & \sum_{i=1}^n \eta_i z_i \leq u \\ & \sum_{i=1}^n \gamma_i z_i + \sum_{i=1}^n b_i y_i \leq v \\ & \frac{1}{2} y_i^2 \leq z_i \qquad i = 1, \dots, n. \end{aligned} \tag{7}$$

Throughout the paper we will assume that (7) admits a strictly feasible solution, i.e., Slater's condition holds for this convex problem. In [1] cases where the optimal values of (7) and (5) are equal are discussed. In this paper we rework and, in some cases, extend the results in [1]. The paper is structured as follows. In Section 2 we discuss the case of a single quadratic constraint and we re-derive in a simpler way the result in [1] for this case. In Section 3 we discuss the case of two quadratic constraints and we derive conditions under which the optimal values of (7) and (5) are equal, which extend those presented in [1]. Finally, in Section 4 we discuss the case with a single trust-region constraint and additional linear constraints. In particular, we focus our attention on the case of two additional linear constraints, which has recently received some attention, see [4,5,14].

Applications of the problems discussed in this paper can be found, e.g., in [1]. In particular, Remark 1 of that paper lists many applications for the case with a single quadratic constraint, while Section 3 extensively discusses many applications for the case of two constraints, like, e.g., the robust formulation of convex quadratic inequalities in presence of ellipsoidal implementation errors. In addition, we observe that: (i) the case of a single constraint includes the well known trust region problem where we additionally have $\mathbf{Q}_1 = \mathbf{I}$ and $\mathbf{q}_1 = \mathbf{0}$ (we refer, e.g., to [9] for a discussion of this problem and some of its generalizations); (ii) the case of two quadratic constraints arises as a subproblem to be solved at each iteration of a trust region algorithm for equality constrained nonlinear problems (see, e.g., [7]); (iii) the case of a quadratic constraint and additional linear constraints arises as a subproblem to be solved at each iteration of a further trust region algorithm for constrained nonlinear problems (see, e.g., [6,12]).

2. The case of a single constraint

The case of a single constraint is a special case of (1) where $\mathbf{Q}_2 = \mathbf{O},\ \mathbf{q}_2 = \mathbf{0},\ v = 0$. In view of (2) we only need to require that $\mathbf{Q}_0$ and $\mathbf{Q}_1$ are SD. After diagonalization, the corresponding relaxed problem is

$$\begin{aligned} \min\quad & \sum_{i=1}^n q_i z_i + \sum_{i=1}^n c_i y_i \\ & \sum_{i=1}^n \eta_i z_i \leq u \\ & \frac{1}{2} y_i^2 \leq z_i \qquad i = 1, \dots, n. \end{aligned} \tag{8}$$

In [1] the following assumption, which removes cases where the problem is unbounded from below, is introduced.

Assumption 2.1. There does not exist any $i \in \{1, \dots, n\}$ such that $q_i, \eta_i \leq 0$ with at least one strict inequality.

In [1] it is proved that the optimal value of the relaxation is always equal to the optimal value of the original problem. The result is proved as follows. Let $(y_i^*, z_i^*),\ i = 1, \dots, n$, be an optimal solution of (8). Let

$$J^* = \left\{ i : z_i^* > \frac{1}{2}\left(y_i^*\right)^2 \right\}.$$

If $J^* = \emptyset$, then it obviously holds that the optimal value of the relaxed problem is equal to that of the original one. In case $J^* \neq \emptyset$, in [1] it is shown how to build a new optimal solution of (8) for which $J^* = \emptyset$. In this section we offer a simplified proof of the result in [1] and prove that, in fact, under very mild assumptions, $J^* = \emptyset$ always holds. To see this, first we derive the KKT conditions for (8), which, under the assumption that Slater's condition holds, allow to compute optimal solutions of this convex problem. The KKT conditions are

$$\begin{aligned} & q_i + \mu\eta_i - \nu_i = 0 && i = 1, \dots, n \\ & c_i + \nu_i y_i = 0 && i = 1, \dots, n \\ & \sum_{i=1}^n \eta_i z_i \leq u \\ & \frac{1}{2} y_i^2 \leq z_i && i = 1, \dots, n \\ & \mu\left(\sum_{i=1}^n \eta_i z_i - u\right) = 0 \\ & \nu_i\left(\frac{1}{2} y_i^2 - z_i\right) = 0 && i = 1, \dots, n \\ & \mu, \nu_i \geq 0 && i = 1, \dots, n, \end{aligned}$$

where μ is the Lagrange multiplier of the constraint $\sum_{i=1}^n \eta_i z_i \leq u$, while $\nu_i,\ i = 1, \dots, n$, are the Lagrange multipliers of the constraints $\frac{1}{2} y_i^2 \leq z_i$. Now, assume that $i \in J^*$. Then, $\nu_i^* = 0$, which can only hold if $c_i = 0$. Thus, we are able to prove the following result.

Theorem 2.1. *There always exists an optimal solution of (8) such that $J^* = \emptyset$.*

Proof. If $c_i \neq 0\ \forall i \in \{1, \dots, n\}$, then the proof immediately follows from the discussion above. A closer inspection of (8) shows that, when $c_i = 0$ for some i, then y_i only appears in the constraint $\frac{1}{2} y_i^2 \leq z_i$, and we can always restrict the attention to optimal solutions where equality holds in such constraint. □

3. The case of two (quadratic) constraints

When we have two constraints, we cannot always guarantee that the optimal value of the relaxed problem (7) is equal to