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Hidden vertices in extensions of polytopes

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ABSTRACT

Some widely known compact extended formulations have the property that each vertex of the corresponding extension polytope is projected onto a vertex of the target polytope. In this paper, we prove that for heptagons with vertices in general position none of the minimum size extensions has this property. Additionally, for any $d \geq 2$ we construct a family of d-polytopes such that at least $\frac{1}{9}$ of all vertices of any of their minimum size extensions is not projected onto vertices.

1. Introduction

The theory of *extended formulations* is a fast developing research field that has connections to many other fields of mathematics. In its core, it deals with the concept of representing polytopes (usually having many facets or even no known linear description) as linear projections of other polytopes (which, preferably, permit smaller linear descriptions). Thus, polyhedral theory plays a crucial role for extended formulations establishing a natural connection to geometry.

Recall that a *polytope* is the convex hull of a finite set of points and that every polytope can be described as the solution set of a system of finitely many linear inequalities and equations. The *size* of a polytope is the number of its facets, i.e., the minimum number of inequalities in a linear description of the polytope. An alternative way to represent a polytope is to write it as a projection of another polytope. Concretely, a polytope $Q \subseteq \mathbb{R}^n$ is called an *extension* of a polytope $P \subseteq \mathbb{R}^d$ if the orthogonal projection of Q on the first d coordinates equals P. The *extension complexity* of a polytope P is the minimum size of any extension for P. Here, we restrict extensions to be polytopes, not polyhedra, as well as projections to be orthogonal, not general affine maps. This definition simplifies the representation, however does not lead to loss of generality.

As an illustration, regular hexagons (having six facets) can be written as a projection of triangular prisms (having only five facets); see Fig. 1. It is easy to argue that such extensions are indeed of minimum size. Another textbook example is the d-dimensional *cross polytope*, which is the convex hull of all unit vectors $\mathbb{e}_i \in \mathbb{R}^d$ and their negatives $-\mathbb{e}_i$ for $i = 1, \ldots, d$. While the d-dimensional cross polytope has 2^d facets, it can be written as the projection onto the x-coordinates of the polytope

$$\left\{(x, y) \in \mathbb{R}^d \times \mathbb{R}^{2d} \;:\; x = \sum_{i=1}^{d} \lambda_i \mathbb{e}_i - \sum_{i=1}^{d} \lambda_{d+i} \mathbb{e}_i, \right.$$
$$\left. \sum_{j=1}^{2d} \lambda_j = 1,\ \lambda_j \geq 0\ \forall j = 1, \ldots, 2d \right\},$$

which is a $(2d-1)$-simplex (and hence has only $2d$ facets) and can also be proven to be of minimum size. Note that both examples admit minimum size extensions whose vertices are again projected to vertices.

In fact, many widely known extended formulations have the property that every vertex of the corresponding extension is projected onto a vertex of the target polytope. See, for instance, extended formulations for the parity polytope [8,2], the permutahedron (as a projection of the Birkhoff-polytope), the cardinality indicating polytope [5], orbitopes [3], or spanning tree polytopes of planar graphs [7]. Although there are not many polytopes whose extension complexity is known exactly, most of the mentioned extensions have minimum size at least up to a constant factor. Moreover, for many of these extensions there is even a one-to-one

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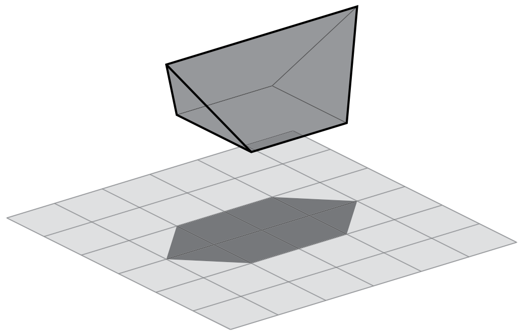

Fig. 1. A hexagon (shadow) as a projection of a triangular prism.

correspondence between the vertices of the extension and the vertices of the target polytope. Clearly, a general extension might have vertices that are not projected onto vertices. Here, let us call such vertices to be *hidden vertices*. The following natural question arises: Given a polytope P, can we always find a minimum size extension Q of P that has no hidden vertices?

In this paper, we negatively answer the above question. Namely, we prove that for almost all heptagons, every minimal extension has at least one hidden vertex. Later we extend this result and construct a family of d-polytopes, $d \geq 2$, such that at least $\frac{1}{9}$ of all vertices in any minimum size extension are hidden.

Thus, in this paper we show that there are polytopes for which the minimum size of an extension without hidden vertices is strictly bigger than their extension complexity. Consider the open question: How big can be the difference between the minimum size of extensions without hidden vertices and the extension complexity? This paper demonstrates that the difference can be at least one in the context of the above question. However, the above question remains open and as the next step to study hidden vertices we propose the following question: Is there a polynomial $q : \mathbb{R} \to \mathbb{R}$ such that for every polytope the minimum size of its extension without hidden vertices is at most $q(s)$, where s is the extension complexity of the polytope?

2. Minimum extensions of heptagons

In this section, we consider convex polygons with seven vertices taken in general position. For such polygons, we prove that there is no extension of minimum size such that every vertex of the extension is projected onto a vertex of the polygon.

2.1. Extension complexity of heptagons

Let us briefly recall known facts about extensions of heptagons. In 2013, Shitov [6] showed that the extension complexity of any convex heptagon is at most 6. Further, it is easy to see that any affine image of a polyhedron with only 5 facets has at most 6 vertices. Thus, one obtains the following theorem.

Theorem 2.1 (*Shitov [6]*). *The extension complexity of any convex heptagon is* 6.

While Shitov's proof is purely algebraic, independently, Padrol and Pfeifle [4] established a geometric proof of this fact. In fact, they showed that any convex heptagon can be written as the projection of a 3-dimensional polytope with 6 facets. In order to get an idea of such a polytope, let us consider the following construction (which is a dual interpretation of the ideas of Padrol and Pfeifle):

Let P be a convex heptagon with vertices $v_1, \dots, v_7$ in cyclic order. For $i \in \{2, 3, 5, 6, 7\}$ let us set $w_i := (v_i, 0) \in \mathbb{R}^3$. Further, choose some numbers $z_1, z_4 > 0$ such that $w_1 := (v_1, z_1)$, $w_4 := (v_4, z_4)$, w_2 and w_3 are contained in one hyperplane and consider $Q' := \text{conv}(\{w_1, \dots, w_7\})$. It can be shown [4] that (by possibly shifting the vertices' labels) one may assume that the convex hull of w_1, w_4 and w_6 forms a facet F of Q'. In this case, remove the defining inequality of F from an irredundant outer description of Q' and obtain a 3-dimensional polytope Q with only 6 facets whose projection is still P. For an illustration, see Fig. 2. Note that removing the facet F results in an additional vertex that projects into the interior of P.

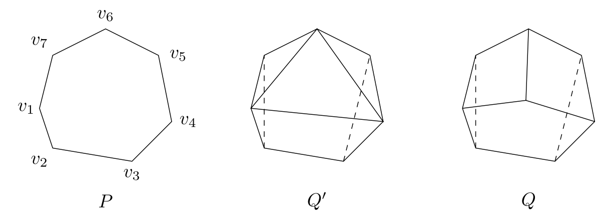

Fig. 2. Example of the construction of a 3-dimensional extension Q with 6 facets for a heptagon P.

In what follows, our argumentation does not rely on the construction described above but only on the statement of Theorem 2.1. However, the previous paragraph gives an intuition why additional vertices may help in order to reduce the number of facets of an extension.

2.2. Additional vertices of minimum size extensions of heptagons

In this section, we will show that most convex heptagons force minimum size extensions to have at least one vertex that is not projected onto a vertex. In order to avoid singular cases in which it is possible to construct minimum size extensions without additional vertices, we only consider convex heptagons P that satisfy the following three conditions:

1. There are no four pairwise distinct vertices $u_1, \dots, u_4$ of P such that the lines $\overline{u_1u_2}$, $\overline{u_3u_4}$ are parallel.
2. There are no six pairwise distinct vertices $u_1, \dots, u_6$ of P, such that the lines $\overline{u_1u_2}$, $\overline{u_3u_4}$, $\overline{u_5u_6}$ have a point common to all three of them.
3. There are no seven pairwise distinct vertices $u_1, \dots, u_7$ of P such that the intersection points $\overline{u_1u_2} \cap \overline{u_3u_4}$, $\overline{u_2u_5} \cap \overline{u_4u_6}$ and $\overline{u_3u_7} \cap \overline{u_1u_5}$ lie in the same line.

Here, a convex heptagon P is called to be *in general position*, if it satisfies conditions (1)–(3). We are now ready to state our main result:

Theorem 2.2. *Let P be a convex heptagon in general position. Then any minimum size extension of P has a vertex that is not projected onto a vertex of P.*

From now on, let us fix a convex heptagon P that is in general position. In order to prove Theorem 2.2, let us assume, for the sake of contradiction, that there exists a polytope Q with only six facets such that Q is an extension of P and every vertex of Q is projected onto a vertex of P. Towards this end, let us first formulate two Lemmas, which we will extensively use through the whole consideration.

Lemma 1. *Let $w_1, \dots, w_4$ be four pairwise distinct vertices of Q such that exactly one pair of them is projected onto the same vertex of P. Then, the dimension of the affine space generated by $w_1, \dots, w_4$ equals* 3.

Proof. Let us assume the contrary and let $w_1, \dots, w_4$ be such vertices of Q that the dimension of the corresponding affine space is at most 2. Then, the dimension of the affine space generated by the projections of $w_1, \dots, w_4$ is at most one since it is the projection of the affine space generated by $w_1, \dots, w_4$, while two distinct points in this space are projected onto the same point. This implies that the projections of $w_1, \dots, w_4$, and thus three different vertices of P, lie on the same line, a contradiction.