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On the distribution-free newsboy problem with some non-skewed demands

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ABSTRACT

We consider three cases of the risk-neutral newsboy problem in which the probability distribution of random demand is only known to be non-skewed with given support, mean and variance. In particular, we derive some closed form formulas for the worst-case and best-case order quantities when this distribution is symmetric, and symmetric and unimodal. Extensions of our results are indicated.

1. Introduction

The classic single-period, single-item inventory problem with random demand, commonly referred to as the newsboy or newsvendor problem, plays a central role at the conceptual foundations of stochastic inventory theory with vast applications in revenue management and supply chain management [18,4,19]. The problem, formulated and solved by Arrow et al. [2] and Morse and Kimball [14], is as follows. Each day the proverbial newsboy has to decide how many newspapers to stock before observing demand. He purchases them from a publisher at a unit cost c and sells them at a price p to customers whose uncertain demand is described by a random variable X. Any unsold items are recycled with a unit salvage value s; to avoid trivialities, $p > c > s$ is assumed. The problem is to find the order (purchase) quantity that maximizes the expected profit. Thus, the decision maker is assumed to be risk-neutral; the models in which he/she is risk-averse, risk-seeking, or uses a maximum entropy approach are proposed in [22,8,1]. Numerous extensions of the newsboy problem were reviewed in [19,13].

Since the demand distribution can be hardly known in practice, Scarf [20] was the first to address the distribution-free newsboy problem, that is, the newsboy problem under incomplete probabilistic information. He assumed that merely the mean $\mu = E(X)$ and the variance $\sigma^2 = \mathrm{Var}(X)$ are known, and proved that whenever $r > \frac{\sigma^2}{\mu^2+\sigma^2}$, where $r = \frac{p-c}{p-s}$, the order quantity $\underline{q}^* = \mu + \frac{(2r-1)\sigma}{2\sqrt{r(1-r)}}$ maximizes the minimum expected profit over all distributions with given μ and σ^2. Thus, Scarf's quantity is the worst-case order quantity. The contribution of Scarf, who is one of the pioneers in inventory theory, was recognized in the 50th anniversary issue of the journal *Operations Research* [21]. The proof of Scarf's formula was simplified and its economic interpretations provided [6]. It was also empirically demonstrated that if the demand distribution is characterized by μ and σ^2, then Scarf's order quantity performs quite well whenever the demand distribution is assumed to be approximately normal [6,7].

The problem of maximizing the expected profit under the best-case demand scenario has been much less examined [6,23,9]. However, it is more trivial because the best-case order quantity is often $\overline{q}^* = \mu$.

Recently, closed form formulas for the worst-case and best-case order quantities were found when the demand distribution has known support $[a, b]$, mean μ, and variance σ^2 [11]. The main purpose of this paper is to derive these two quantities under the assumption that the distribution is non-skewed, symmetric, or symmetric and unimodal, with given support, mean, and variance.

The paper is organized as follows. In Section 2 we formulate the problem under study, and in Section 3 we present the theoretical background for seeking the order quantities under the worst-case and best case scenarios. These quantities are listed in Section 4. Final remarks, including future research and some extensions of the obtained results, are made in Section 5.

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2. Problem formulation

If q is an order quantity and X denotes the random demand, $\min(X, q)$ represents the demand that is met and $q - \min(X, q)$ is the salvage amount. Consequently, the expected profit is expressed by

$$\begin{aligned}\pi(q) &= pE[\min(X,q)] + sE[q - \min(X,q)] - cq \\ &= (p-s)E[\min(X,q)] - (c-s)q \\ &= (p-s)\int_{-\infty}^{\infty} \min(x,q)\,dF(x) - (c-s)q,\end{aligned}$$

where F is the cdf (cumulative distribution function) of X.

Since for every cdf F (defined as a right-continuous function) with a finite mean

$$\frac{\partial_+}{\partial q}\left[\int_{-\infty}^{\infty} \min(x,q)\,dF(x)\right] = \frac{\partial_+}{\partial q}\left[q - \int_{-\infty}^{q} F(x)\,dx\right] = 1 - F(q),$$

the first right derivative of $\pi(q)$ is $\frac{\partial_+}{\partial q}[\pi(q)] = (p-c) - (p-s)F(q)$, and the optimal order quantity q^* is typically defined as the smallest q such that $F(q) \geq r$. Note that $q^* = F^{-1}(r)$ whenever X is a continuous random variable.

Suppose only a partial information about the cdf F of X is available in the sense that $F \in \mathcal{F}$, where $\mathcal{F}$ is a non-empty family of cdfs representing some distributions bounded on $[a, b]$. For every $q \in [a, b]$, let $L(q)$ and $U(q)$ be sharp lower and upper bounds on the expected met demand $E[\min(X, q)]$, that is, there exist $\underline{F}_q, \overline{F}_q \in \mathcal{F}$ such that

$$L(q) = \int_a^b \min(x,q)\,d\underline{F}_q(x) = \min_{F\in\mathcal{F}} \int_a^b \min(x,q)\,dF(x),$$

$$U(q) = \int_a^b \min(x,q)\,d\overline{F}_q(x) = \max_{F\in\mathcal{F}} \int_a^b \min(x,q)\,dF(x).$$

The bounds $L(q)$ and $U(q)$ lead to the following sharp lower and upper bounds on the expected profit $\pi(q)$:

$$\underline{\pi}(q) = (p-s)L(q) - (c-s)q \quad \text{and}$$
$$\overline{\pi}(q) = (p-s)U(q) - (c-s)q.$$

When $\underline{\pi}(q)$ and $\overline{\pi}(q)$ are maximized over q, one can find the worst-case and best-case order quantities denoted by $\underline{q}^*$ and $\overline{q}^*$, respectively. Consequently, for any $F \in \mathcal{F}$, the corresponding maximum expected profit $\pi(q^*)$ satisfies $\underline{\pi}\left(\underline{q}^*\right) \leq \pi(q^*) \leq \overline{\pi}\left(\overline{q}^*\right)$.

In this paper we assume that the mean demand is $\mu = (a+b)/2$, that is, the allowable distributions are on the interval $[\mu-d, \mu+d]$, where $0 < d \leq \mu$. Under this assumption, we consider the cases when these distributions are additionally non-skewed, symmetric, and symmetric and unimodal.

3. Theoretical background

As it was observed in the previous section, for a given non-empty family $\mathcal{F}$ of cdfs on $[a, b]$, to find the worst-case and best-case order quantities, $\underline{q}^*$ and $\overline{q}^*$, it suffices to identify sharp bounds, $L(q)$ and $U(q)$, on the expected met demand $E[\min(X, q)]$. If $L(q)$ ($U(q)$) is concave, then $\underline{F}(q) = 1 - \frac{\partial_+ L(q)}{\partial q}$ ($\overline{F}(q) = 1 - \frac{\partial_+ U(q)}{\partial q}$) is the infimum (supremum) of $\mathcal{F}$ with respect to the increasing concave order, and $\underline{q}^*$ ($\overline{q}^*$) is the smallest q such that $\underline{F}(q) \geq r$ ($\overline{F}(q) \geq r$) [11]. Recall here that F is said to be smaller than G in the sense of this order, written $F \preccurlyeq_{icv} G$, if for every non-decreasing and concave function $\varphi(x)$, $\int_a^b \varphi(x)\,dF(x) \leq \int_a^b \varphi(x)\,dG(x)$. Furthermore, $F \preccurlyeq_{icv} G$ is equivalent to $\int_a^b \min(x,q)dF(x) \leq \int_a^b \min(x,q)dG(x)$ for every $q \in [a, b]$. A cdf $\underline{F}$ ($\overline{F}$) is called the infimum (supremum) of F with respect to $\preccurlyeq_{icv}$ if $\underline{F}$ ($\overline{F}$) is the greatest (smallest) cdf, not necessarily in $\mathcal{F}$, such that $\underline{F} \preccurlyeq_{icv} F$ ($F \preccurlyeq_{icv} \overline{F}$) for all $F \in \mathcal{F}$ [15].

Let δ_x and $U_{\mu\mp x}$ denote the cdfs of the one-point (degenerate) distribution at x, and the uniform distribution on $[\mu - x, \mu + x]$, respectively. It is well-known that if $\mathcal{F}$ represents all distributions (unimodal distributions) on $[\mu - d, \mu + d]$, then $\frac{1}{2}\delta_{\mu-d} + \frac{1}{2}\delta_{\mu+d}$ ($U_{\mu\mp d}$) and δ_μ are the minimum and maximum of $\mathcal{F}$. Since $\frac{1}{2}\delta_{\mu-d} + \frac{1}{2}\delta_{\mu+d}$ and $U_{\mu\mp d}$ have variances d^2 and $d^2/3$, respectively, the following is true; see e.g. [5,3].

Lemma 1. *Let X have a distribution on $[\mu - d, \mu + d]$. Then its variance σ^2 satisfies $\sigma^2 \leq d^2$, and the bound d^2 remains sharp when the distribution is symmetric. If it is also unimodal, then $\sigma^2 \leq d^2/3$.*

Lemma 1 will clarify the assumptions imposed on the variance in the remainder of this section.

First, we assume that the demand distributions on $[\mu-d, \mu+d]$ (with mean μ) are non-skewed, that is, $E\left[(X-\mu)^3\right] = 0$. The next lemma can be deduced from Theorem 2.1 in the excellent monograph of Karlin and Studden [12, p. 472]; see also Theorem 3.1 [17] and Theorem 2 [11].

Lemma 2. *Let X have a non-skewed distribution on $[\mu - d, \mu + d]$ with variance $\sigma^2 < d^2$. Then for every $q \in [\mu - d, \mu + d]$, the sharp lower and upper bounds, $L(q)$ and $U(q)$, on $E[\min(X, q)]$ can be defined as follows:*

$$L(q) = \max_{c_i}\left[c_0 + \mu c_1 + \left(\mu^2+\sigma^2\right)c_2 + \left(\mu^3 + 3\mu\sigma^2\right)c_3\right]$$

$$\text{s.t. } c_0 + c_1x + c_2x^2 + c_3x^3 \leq \min(x,q) \quad \text{for } x \in [\mu-d, \mu+d];$$

$$U(q) = \min_{c_i}\left[c_0 + \mu c_1 + \left(\mu^2+\sigma^2\right)c_2 + \left(\mu^3 + 3\mu\sigma^2\right)c_3\right]$$

$$\text{s.t. } c_0 + c_1x + c_2x^2 + c_3x^3 \geq \min(x,q) \quad \text{for } x \in [\mu-d, \mu+d].$$

Theorem 1. *Let the cdf of X belong to the family $\mathcal{F}$ representing the non-skewed distributions on $[\mu - d, \mu + d]$ with variance $\sigma^2 < d^2$. Then the sharp lower and upper bounds on $E[\min(X, q)]$ are:*

$$L(q) = \begin{cases} \min(\mu,q) - \dfrac{(d-|\mu-q|)\,\sigma^2}{2d^2} & \text{if } \dfrac{3d}{5} \leq |\mu-q| \leq d, \\ \min(\mu,q) - \dfrac{(z-|\mu-q|)\left(d^2-\sigma^2\right)\sigma^2}{(z+d)\left(dz^2 - 2\sigma^2 z + d\sigma^2\right)} & \\ \quad \text{if } \dfrac{\sigma^2}{d+2\mu} \leq |\mu-q| \leq \dfrac{3d}{5}, & \\ \dfrac{q+\mu-\sigma}{2} & \text{if } |\mu-q| \leq \dfrac{\sigma^2}{d+2\sigma}, \end{cases}$$

where $z = y^* - d$,

$$y^* = \begin{cases} \dfrac{1}{6d}\left[B_1 + B_2 + 2(B_1 - B_2)\cos\left(\dfrac{1}{3}\operatorname{arcos}(1+R)\right)\right] & \\ \quad \text{if } |\mu-q| \geq e-d, & \\ \dfrac{1}{6d}\left[B_1 + B_2 + 2(B_1 - B_2)\cosh\left(\dfrac{1}{3}\operatorname{arcosh}(1+R)\right)\right] & \\ \quad \text{otherwise}, & \end{cases}$$

$$B_1 = 3d(|\mu-q| + d), \qquad B_2 = 2(d^2+\sigma^2),$$