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On upper bounds for assortment optimization under the mixture of multinomial logit models

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ABSTRACT

The assortment optimization problem under the mixture of multinomial logit models is NP-complete and there are different approximation methods to obtain upper bounds on the optimal expected revenue. In this paper, we analytically compare the upper bounds obtained by the different approximation methods. We propose a new, tractable approach to construct an upper bound on the optimal expected revenue and show that it obtains the tightest bound among the existing tractable approaches in the literature to obtain upper bounds.

Assortment optimization has important applications in retailing and revenue management and has received much attention lately. In the assortment problem, we have a firm that is interested in maximizing revenues by selling products to customers, where each product has a revenue associated with it and customers choose among the offered products according to a given discrete choice model. The goal therefore is to figure out the set of products, or the assortment, that maximizes the expected revenue obtained from a customer. While there are a large number of discrete choice models that can be used to describe customer choice behavior, the multinomial logit model and its variants have been a popular choice in the assortment optimization literature.

In this paper, we consider the assortment problem under a mixture of multinomial logit models. In this model, we have multiple customers segments and an arriving customer belongs to a particular segment with a given probability and chooses among the offered products according to the multinomial logit model. The parameters of the multinomial logit model are allowed to depend on the segment to which the customer belongs.

The assortment optimization problem under the mixture of multinomial logit models is NP-complete; see [1,8]. On the other hand, [6] shows that the mixture of multinomial logits is a rich choice model that can approximate any random utility choice model arbitrarily closely. So there has been considerable interest in the assortment problem under the mixture of multinomial logit models and there is a growing literature that focuses on developing approximation methods that generate assortments with provable performance guarantees; see for example [8,7].

Another interesting research direction is to obtain upper bounds on the optimal assortment revenue, since upper bounds are useful in getting a better handle on the revenue performance gaps of the candidate assortments obtained by the different approximation methods. [1] formulates the assortment problem under the mixture of multinomial logit models as a linear mixed integer program, the linear programming relaxation of which gives an upper bound on the optimal expected revenue. [2] proposes a Lagrangian relaxation approach where they relax the constraints that the same assortment be offered to the different customer segments by associating Lagrange multipliers with them. While the Lagrangian relaxation approach obtains an upper bound, solving it turns out to be intractable. Therefore the authors propose a further approximation that is based on solving a continuous knapsack problem over a discrete set of grid points. They show that this approximation method is tractable and continues to provide an upper bound on the optimal expected revenue. One difficulty with the approximation method proposed in [2] is that the quality of the upper bound depends on the density of the grid, which makes it challenging to establish analytical results.

In this paper, we focus on solution methods to obtain upper bounds on the optimal expected revenue for the assortment problem under the mixture of multinomial logit models. We first provide a partial characterization of the optimal assortment. In particular, we show that an optimal assortment includes a certain revenue ordered subset of the products, and this subset can be obtained efficiently. This structure can be potentially exploited to

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speed up the solution methods by reducing the size of the search space. We then analytically compare the upper bounds obtained by the different solution methods in the literature. We show that the Lagrangian relaxation approach obtains the tightest bound among the available approaches to obtain upper bounds. However, since the Lagrangian relaxation is NP-complete, we propose a new, alternative approach to obtain an upper bound on the optimal expected revenue. Our approach builds on the ideas developed in [2,5]. The Lagrangian relaxation approach in [2] decomposes the assortment problem involving multiple customer segments into a number of single segment problems where there is a fixed cost of introducing a product into the assortment. [5] studies the assortment problem under the multinomial logit model when there is a fixed cost of introducing a product into the assortment and propose tractable solution methods. While our approach is a simple adaptation of the ideas developed in the above mentioned papers, it yields a solution method that is tractable and provably tighter than the other tractable approaches in the literature to obtain upper bounds on the optimal expected revenue.

In summary, we make the following contributions in this paper. (1) We provide a partial characterization of the solution to the assortment problem under the mixture of multinomial logit models. (2) We analytically compare the different methods proposed in the literature to obtain upper bounds on the optimal expected revenue. We show that the Lagrangian relaxation approach proposed in [2] obtains the tightest upper bound. However, the Lagrangian relaxation cannot be solved efficiently. (3) We propose a new approach to obtain an upper bound that remains tractable. The bound obtained by our approach is provably tighter than the other tractable solution methods in the literature to obtain upper bounds. Moreover, in contrast to [2], our solution method does not require a discrete set of grid points and thus reduces some of the subjectivity involved in choosing a discretization scheme. (4) Computational experiments indicate that our approach can be beneficial both in terms of tighter bounds and faster run times.

The rest of the paper is organized as follows. In Section 1, we describe the assortment problem under the mixture of multinomial logit models. In Section 2, we describe the solution methods proposed in the literature to obtain upper bounds on the optimal expected revenue. We analytically compare the upper bounds obtained by the proposed methods in Section 3. In Section 4, we describe our solution approach and show that it obtains the tightest bound among the tractable solution methods in the literature to obtain upper bounds on the optimal expected revenue. Section 5 presents our computational experiments.

1. Problem formulation

We consider the assortment optimization problem under the mixture of multinomial logit models. We have N products and the revenue associated with product $j \in \{1, \dots, N\}$ is p_j. We let $x_j \in \{0, 1\}$ denote whether we offer product j. We have L customer segments interested in purchasing a product from the offered assortment. The preference weight for customer segment $l \in \{1, \dots, L\}$ for product j is v_j^l, while the preference weight associated with a segment l customer not purchasing anything is normalized to be 1. Within each segment, choice is governed by the multinomial logit model and so the probability that a segment l customer purchases product j is $v_j^l x_j/(1 + \sum_k v_k^l x_k)$. We let α^l denote the arrival rate of customer segment l. The assortment problem is to decide which products to make available to an arriving customer in order to maximize the expected total revenue. The optimal assortment can be obtained by solving the problem

$$Z^{OPT} = \max_{x|x_j\in\{0,1\}} \sum_l \alpha^l \frac{\sum_j p_j v_j^l x_j}{\sum_j v_j^l x_j + 1}. \tag{1}$$

Solving assortment problem (1) and obtaining the optimal expected revenue is intractable; see [1,8]. Therefore, it is unlikely that we can provide a complete characterization of the optimal solution to problem (1). However, Lemma 1 below provides a partial characterization of the structure of an optimal solution.

We begin with some preliminaries. Assume without loss of generality that the products are indexed in order of decreasing revenues so that $p_1 \ge p_2 \ge \cdots \ge p_N$, and consider the assortment problem involving customer segment l alone

$$\max_{x^l|x_j^l\in\{0,1\}} \frac{\sum_j p_j v_j^l x_j^l}{\sum_j v_j^l x_j^l + 1}. \tag{2}$$

It is known that revenue ordered assortments are optimal for problem (2); see for example [4]. That is, there exists an optimal solution $\hat{x}^l = \{\hat{x}_j^l|\forall j\}$ to problem (2) with $\hat{x}_j^l = 1$ for $j \in \{1, \dots, J^l\}$ and $\hat{x}_j^l = 0$ for $j \in \{J^l + 1, \dots, N\}$, where $J^l \in \{1, \dots, N\}$. We have the following lemma.

Lemma 1. *Let $\hat{J} = \min_l\{J^l\}$. There exists an optimal solution $\hat{x} = \{\hat{x}_j|\forall j\}$ to problem (1) with $\hat{x}_j = 1$ for $j \in \{1, \dots, \hat{J}\}$.*

Proof. Suppose that the statement of the lemma is false. Let $\hat{x}$ be an optimal solution to problem (1) and let $\kappa \in \{1, \dots, \hat{J}\}$ be the smallest index such that $\hat{x}_\kappa = 0$. We let $\tilde{x} = \{\tilde{x}_j|\forall j\}$ be the same as $\hat{x}$ except that $\tilde{x}_\kappa = 1$. We show below that the solution $\tilde{x}$ generates as much revenue as $\hat{x}$, and is therefore also optimal.

Let $S = \{1, \dots, N\} \setminus \{\kappa\}$ and $T = \{1, \dots, \kappa - 1\}$. Fix a segment l and note that

$$\frac{\sum_{j\in S} p_j v_j^l \hat{x}_j}{\sum_{j\in S} v_j^l \hat{x}_j + 1} = \left(\frac{\sum_{j\in T} v_j^l \hat{x}_j + 1}{\sum_{j\in S} v_j^l \hat{x}_j + 1}\right) \frac{\sum_{j\in T} p_j v_j^l \hat{x}_j}{\sum_{j\in T} v_j^l \hat{x}_j + 1} + \left(\frac{\sum_{j\in S\setminus T} v_j^l \hat{x}_j}{\sum_{j\in S} v_j^l \hat{x}_j + 1}\right) \frac{\sum_{j\in S\setminus T} p_j v_j^l \hat{x}_j}{\sum_{j\in S\setminus T} v_j^l \hat{x}_j}. \tag{3}$$

Since $\kappa \in \{1, \dots, \jmath\}$ and $\jmath \le J^l$, it follows that $T \subsetneq \{1, \dots, J^l\}$, the optimal revenue ordered assortment for segment l. Lemma 2 in [4] then implies that $\frac{\sum_{j\in T} p_j v_j^l \hat{x}_j}{\sum_{j\in T} v_j^l \hat{x}_j + 1} \le \frac{\sum_{j\in T} p_j v_j^l \hat{x}_j + p_\kappa v_\kappa^l}{\sum_{j\in T} v_j^l \hat{x}_j + v_\kappa^l + 1}$. This together with Lemma 3.1 in [9] implies that $p_\kappa \ge \frac{\sum_{j\in T} p_j v_j^l \hat{x}_j}{\sum_{j\in T} v_j^l \hat{x}_j + 1}$. On the other hand, since the products are revenue ordered, $p_\kappa \ge p_j$ for all $j \in S \setminus T$ and we have $p_\kappa \ge \frac{\sum_{j\in S\setminus T} p_j v_j^l \hat{x}_j}{\sum_{j\in S\setminus T} v_j^l \hat{x}_j}$. The above statements together with (3) imply that

$$\frac{\sum_{j\in S} p_j v_j^l \hat{x}_j}{\sum_{j\in S} v_j^l \hat{x}_j + 1} \le p_\kappa. \tag{4}$$

Now consider the difference in the expected revenues from segment l under $\tilde{x}$ and $\hat{x}$. We have

$$\frac{\sum_j p_j v_j^l \tilde{x}_j}{\sum_j v_j^l \tilde{x}_j^l + 1} - \frac{\sum_j p_j v_j^l \hat{x}_j}{\sum_j v_j^l \hat{x}_j + 1}$$

$$= \sum_{j\in S} p_j v_j^l \hat{x}_j \left[\frac{1}{\sum_{j\in S} v_j^l \hat{x}_j + v_\kappa^l + 1} - \frac{1}{\sum_{j\in S} v_j^l \hat{x}_j + 1}\right]$$