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On likelihood ratio ordering of parallel system with two exponential components

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ABSTRACT

This paper considers stochastic comparison for parallel systems with two exponential components. For a given such system, we identify a region, such that, if the hazard rate pair of another parallel system lies in that region, then there exists likelihood ratio ordering between the two systems. The new results presented in this paper extend most existing ones in the literature.

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1. Introduction

Parallel systems are commonly used to improve the reliability of a device. For economical reason, parallel systems with only two components are quite popular. To compare the lifetimes of two such systems is fundamental in engineering reliability theory.

For systems made up of components with general distributed lifetimes, the stochastic comparison for those systems is quite elusive, since the distribution theory becomes quite complicated. For this reason, in this paper, we confine ourselves to the parallel systems with components whose lifetimes follow exponential distributions. Pledger and Proschan [11] were the first ones to compare stochastically for such systems. Since then, many researchers have worked in this field, including Kochar and Rojo [7], Dykstra et al. [3], Khaledi and Kochar [5,6], Kochar and Xu [8], Joo and Mi [4], Da et al. [2], and Zhao and Balakrishnan [14].

Let $T(\lambda_1, \ldots, \lambda_n)$ be the lifetime of a parallel system with n exponential components whose hazard rates are $\lambda_1, \ldots, \lambda_n$, respectively. For technical reason, we just focus on the case of $n = 2$ in the present paper. For the systems $T(\lambda_1, \lambda_2)$ and $T(\gamma_1, \gamma_2)$, by symmetry, we assume $\lambda_1 \le \lambda_2$ and $\gamma_1 \le \gamma_2$. Denote by $\succ$ as majorization order, $\ge_{st}$, $\ge_{hr}$, $\ge_{rh}$, and $\ge_{lr}$ as the usual stochastic order, the hazard rate order, the reversed hazard rate order, and the likelihood ratio order, respectively. The formal definitions of these orders will be given in the next section.

So far, several stochastic comparison results for the lifetimes $T(\lambda_1, \lambda_2)$ and $T(\gamma_1, \gamma_2)$ have been established. For instance, Pledger and Proschan [11] showed that if $(\lambda_1, \lambda_2) \succ (\gamma_1, \gamma_2)$, $T(\lambda_1, \lambda_2) \ge_{st} T(\gamma_1, \gamma_2)$. Boland et al. [1] strengthened it as $T(\lambda_1, \lambda_2) \ge_{hr} T(\gamma_1, \gamma_2)$ and $T(\lambda_1, \lambda_2) \ge_{rh} T(\gamma_1, \gamma_2)$. Dykstra et al. [3] further enhanced it as $T(\lambda_1, \lambda_2) \ge_{lr} T(\gamma_1, \gamma_2)$. Joo and Mi [4] revealed that when $\lambda_1 \le \gamma_1 \le \gamma_2 \le \lambda_2$ and $\lambda_1 + \lambda_2 \le \gamma_1 + \gamma_2$, $T(\lambda_1, \lambda_2) \ge_{hr} T(\gamma_1, \gamma_2)$. Zhao and Balakrishnan [14] improved this result from hazard rate order to likelihood ratio order. Yan et al. [13] showed, when $\lambda_1 \le \gamma_1 \le \lambda_2 \le \gamma_2$ and $\lambda_1 + \gamma_2 \le \gamma_1 + \lambda_2$, $T(\lambda_1, \lambda_2) \ge_{lr} T(\gamma_1, \gamma_2)$. Misra and Misra [9] proved that, when (λ_1, λ_2) weakly majorizes (γ_1, γ_2), then $T(\lambda_1, \lambda_2) \ge_{rh} T(\gamma_1, \gamma_2)$.

For a given point (λ_1, λ_2), denote $\Omega_{\lambda_1,\lambda_2}$ as the region such that, if $(\gamma_1, \gamma_2) \in \Omega_{\lambda_1,\lambda_2}$, (γ_1, γ_2) is weakly majorized by (λ_1, λ_2). The result of Misra and Misra [9] thus can be stated as: when $(\gamma_1, \gamma_2) \in \Omega_{\lambda_1,\lambda_2}$, $T(\lambda_1, \lambda_2) \ge_{rh} T(\gamma_1, \gamma_2)$.

A natural question now is: can we extend the result of Misra and Misra [9] to likelihood ratio order? Or, in what a subregion of $\Omega_{\lambda_1,\lambda_2}$ that $T(\lambda_1, \lambda_2) \ge_{lr} T(\gamma_1, \gamma_2)$ holds?

This paper intends to answer this question. In this paper, we reveal a subregion of $\Omega_{\lambda_1,\lambda_2}$, denote it as $\Theta_{\lambda_1,\lambda_2}$, such that, for any $(\gamma_1, \gamma_2) \in \Theta_{\lambda_1,\lambda_2}$, $T(\lambda_1, \lambda_2) \ge_{lr} T(\gamma_1, \gamma_2)$. The picture below shows the regions $\Omega_{\lambda_1,\lambda_2}$ and $\Theta_{\lambda_1,\lambda_2}$, where $\bar{\lambda} = (\lambda_1 + \lambda_2)/2$.

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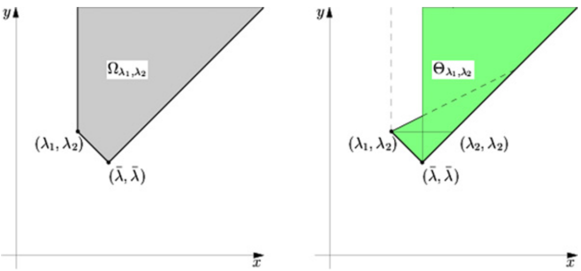

The region $\Omega_{\lambda_1,\lambda_2}$ is formed by the inequalities $y \geq x \geq \lambda_1$, and $x + y \geq \lambda_1 + \lambda_2$, and the region $\Theta_{\lambda_1,\lambda_2}$ is formed by the inequalities $y \geq x \geq \bar{\lambda}$, or, $x + y \geq \lambda_1 + \lambda_2$, and $y \leq \lambda_2 + \frac{1}{2}(x - \lambda_1)$. Notice $(\lambda_1, \lambda_2) \succ (\gamma_1, \gamma_2)$ is equivalent to that the point (γ_1, γ_2) is on the line segment connecting the point (λ_1, λ_2) and $(\bar{\lambda}, \bar{\lambda})$. Thus, Dykstra's result can be stated as: for any point (γ_1, γ_2) on that line segment, $T(\lambda_1, \lambda_2) \geq_{lr} T(\gamma_1, \gamma_2)$. The result of Zhao and Balakrishnan [14] can be stated as: for any point (γ_1, γ_2) in the triangle formed by the three points (λ_1, λ_2), $(\bar{\lambda}, \bar{\lambda})$, and (λ_2, λ_2), $T(\lambda_1, \lambda_2) \geq_{lr} T(\gamma_1, \gamma_2)$, and the result of Yan et al. [13] can be stated as: for any point (γ_1, γ_2) lies in the triangle with vertexes (λ_1, λ_2), (λ_2, λ_2), and $(\lambda_2, 2\lambda_2 - \lambda_1)$, $T(\lambda_1, \lambda_2) \geq_{lr} T(\gamma_1, \gamma_2)$. As we can see, the result established in this paper extends the results of Dykstra et al. [3], Zhao and Balakrishnan [14], improves part of the result of Da et al. [2] from hazard ratio ordering to likelihood ratio ordering, and overlaps that of Yan et al. [13].

2. Definitions and notations

To state the results, we introduce some notations and concepts first. Let X be a nonnegative continuous random variable with distribution function $F_X(t)$, survival function $\bar{F}_X(t) = 1 - F_X(t)$, and density function $f_X(t)$. The hazard function and the reversed hazard function of X are defined as $\lambda_X = f_X/\bar{F}_X$ and $r_X = f_X/F_X$, respectively.

For two nonnegative continuous random variables X and Y, X is said to be smaller than Y in the usual stochastic order (denoted by $X \leq_{st} Y$), if $\bar{F}_X(t) \leq \bar{F}_Y(t)$. X is said to be smaller than Y in hazard rate order (denoted by $X \leq_{hr} Y$), if $\lambda_X(t) \geq \lambda_Y(t)$. X is said to be smaller than Y in reversed hazard rate order (denoted by $X \leq_{rh} Y$), if $r_X(t) \leq r_Y(t)$. X is said to be smaller than Y in likelihood ratio order (denoted by $X \leq_{lr} Y$), if the ratio $f_Y(t)/f_X(t)$ is increasing in t. It is well known that likelihood ratio order implies both hazard rate order and reversed hazard rate order, while these two orders imply usual stochastic order.

Given two vectors $\boldsymbol{a} = (a_1, a_2, \ldots, a_n)$ and $\boldsymbol{b} = (b_1, b_2, \ldots, b_n)$, let $a_{(1)} \leq a_{(2)} \leq \cdots \leq a_{(n)}$ and $b_{(1)} \leq b_{(2)} \leq \cdots \leq b_{(n)}$ be the increasing arrangements of the components of the two vectors, then the vector $\boldsymbol{a}$ is said to majorize the vector $\boldsymbol{b}$ (denoted by $\boldsymbol{a} \succ \boldsymbol{b}$) if and only if, $\sum_{i=1}^{n} a_{(i)} = \sum_{i=1}^{n} b_{(i)}$, and $\sum_{i=1}^{k} a_{(i)} \leq \sum_{i=1}^{k} b_{(i)}$, for $k = 1, \ldots, n-1$. If for $k = 1, \ldots, n$, $\sum_{i=1}^{k} a_{(i)} \leq \sum_{i=1}^{k} b_{(i)}$, then the vector $\boldsymbol{a}$ is said to weakly majorize the vector $\boldsymbol{b}$ (denoted by $\boldsymbol{a} \overset{w}{\succ} \boldsymbol{b}$). For some extensive and comprehensive discussions on the theory of these orders and their applications, one can see Müller and Stoyan [10], or, Shaked and Shanthikumar [12].

3. Main result and proof

We state the main result as the following theorem.

Theorem 3.1. *For any $(\gamma_1, \gamma_2) \in \Theta_{\lambda_1,\lambda_2}$, $T(\lambda_1, \lambda_2) \geq_{lr} T(\gamma_1, \gamma_2)$.*

To prove the theorem, the following lemmas will be used. The proof of the first lemma is easy and thus is omitted.

Lemma 3.2. *For $x > 0$, the function $f(x) = \frac{x}{1-e^{-x}}$ is increasing and convex.*

The following lemma is a classical result of Dykstra et al. [3]:

Lemma 3.3 (*Theorem 3.1, Dykstra et al. [3]*).

If $(\lambda_1, \lambda_2) \succ (\gamma_1, \gamma_2)$, *then*, $T(\lambda_1, \lambda_2) \geq_{lr} T(\gamma_1, \gamma_2)$.

Proof of Theorem 3.1. At first, by using Theorem 1. C. 33 in Shaked and Shanthikumar [12], we have, when $\gamma_1 \geq \bar{\lambda}$, $T(\lambda_1, \lambda_2) \geq_{lr} T(\gamma_1, \gamma_2)$. Now we want to show $T(\lambda_1, \lambda_2) \geq_{lr} T(\gamma_1, \gamma_2)$ for all point (γ_1, γ_2) in the triangle region formed by the lines $x + y = \lambda_1 + \lambda_2$, $y = x$, and $y = \lambda_2 + \frac{1}{2}(x - \lambda_1)$.

By the result of Misra and Misra [9], $T(\lambda_1, \lambda_2) \geq_{rh} T(\gamma_1, \gamma_2)$ on that region. From Theorem 1.C.4(b) of Shaked and Shanthikumar [12], it is enough to show that the ratio of the reversed hazard rate functions is increasing for $t > 0$.

Denote the reversed hazard rate function of $T(\lambda_1, \lambda_2)$ as $r_{(\lambda_1,\lambda_2)}(t)$. We have,

$$r_{(\lambda_1,\lambda_2)}(t) = \frac{\lambda_1 e^{-\lambda_1 t}(1 - e^{-\lambda_2 t}) + \lambda_2 e^{-\lambda_2 t}(1 - e^{-\lambda_1 t})}{(1 - e^{-\lambda_1 t})(1 - e^{-\lambda_2 t})} = \frac{\lambda_1 e^{-\lambda_1 t}}{1 - e^{-\lambda_1 t}} + \frac{\lambda_2 e^{-\lambda_2 t}}{1 - e^{-\lambda_2 t}}.$$

For our convenience, we denote $A \overset{\text{sgn}}{=} B$ if the signs of A and B are the same. We have,

$$\psi(t) = \frac{r_{(\lambda_1,\lambda_2)}(t)}{r_{(\gamma_1,\gamma_2)}(t)} = \frac{\frac{\lambda_1 e^{-\lambda_1 t}}{1-e^{-\lambda_1 t}} + \frac{\lambda_2 e^{-\lambda_2 t}}{1-e^{-\lambda_2 t}}}{\frac{\gamma_1 e^{-\gamma_1 t}}{1-e^{-\gamma_1 t}} + \frac{\gamma_2 e^{-\gamma_2 t}}{1-e^{-\gamma_2 t}}} \overset{\text{def}}{=} \frac{\varphi(\lambda_1, \lambda_2; t)}{\varphi(\gamma_1, \gamma_2; t)},$$

so,

$$\psi'(t) \overset{\text{sgn}}{=} \varphi_t'(\lambda_1, \lambda_2; t)\varphi(\gamma_1, \gamma_2; t) - \varphi(\lambda_1, \lambda_2; t)\varphi_t'(\gamma_1, \gamma_2; t) \overset{\text{sgn}}{=} \frac{\varphi_t'(\lambda_1, \lambda_2; t)}{\varphi(\lambda_1, \lambda_2; t)} - \frac{\varphi_t'(\gamma_1, \gamma_2; t)}{\varphi(\gamma_1, \gamma_2; t)},$$

where

$$\frac{\varphi_t'(\lambda_1, \lambda_2; t)}{\varphi(\lambda_1, \lambda_2; t)} = -\frac{\lambda_1^2 e^{-\lambda_1 t}(1 - e^{-\lambda_2 t})^2 + \lambda_2^2 e^{-\lambda_2 t}(1 - e^{-\lambda_1 t})^2}{\lambda_1 e^{-\lambda_1 t}(1 - e^{-\lambda_1 t})(1 - e^{-\lambda_2 t})^2 + \lambda_2 e^{-\lambda_2 t}(1 - e^{-\lambda_2 t})(1 - e^{-\lambda_1 t})^2},$$

and similarly to $\varphi_t'(\gamma_1, \gamma_2; t)/\varphi(\gamma_1, \gamma_2; t)$.

Consider the function

$$\Psi(x_1, x_2) = \frac{x_1^2 e^{-x_1}(1 - e^{-x_2})^2 + x_2^2 e^{-x_2}(1 - e^{-x_1})^2}{x_1 e^{-x_1}(1 - e^{-x_1})(1 - e^{-x_2})^2 + x_2 e^{-x_2}(1 - e^{-x_2})(1 - e^{-x_1})^2}, \quad 0 < x_1 \leq x_2.$$

To show $\psi'(t) \geq 0$ for all (γ_1, γ_2) in that triangle region, we just need to verify that the function $\Psi(x_1, x_2)$ is increasing along the direction $\boldsymbol{v} = (1, \alpha)$ with $0 \leq \alpha \leq 1/2$.

Denote the numerator part of $\Psi(x_1, x_2)$ as N, and the denominator part as D. We have,

$$\nabla_{\boldsymbol{v}} \Psi = \left(\frac{\partial \Psi}{\partial x_1}, \frac{\partial \Psi}{\partial x_2}\right)(1, \alpha) \overset{\text{sgn}}{=} \left(\frac{\partial N}{\partial x_1} D - N\frac{\partial D}{\partial x_1}, \frac{\partial N}{\partial x_2} D - N\frac{\partial D}{\partial x_2}\right)(1, \alpha).$$

Some calculations lead,

$$\frac{\partial N}{\partial x_1} = (2x_1 - x_1^2)e^{-x_1}(1 - e^{-x_2})^2 + 2x_2^2 e^{-(x_1+x_2)}(1 - e^{-x_1}),$$

$$\frac{\partial D}{\partial x_1} = g(x_1)e^{-x_1}(1 - e^{-x_2})^2 + 2x_2 e^{-(x_1+x_2)}(1 - e^{-x_1})(1 - e^{-x_2}),$$