Operations Research Letters 43 (2015) 199–202

Contents lists available at ScienceDirect

Operations Research Letters

journal homepage: www.elsevier.com/locate/orl

On the lattice programming gap of the group problems

Iskander Aliev

School of Mathematics, Cardiff University, Cardiff, Wales, UK

ARTICLE INFO

Article history:
Received 21 April 2014
Received in revised form
17 December 2014
Accepted 19 January 2015
Available online 2 February 2015

Keywords:
Group relaxations
Integer programming gap
Lattices
Diameters of graphs
Covering radius
Frobenius numbers

ABSTRACT

Given a full-dimensional lattice $\Lambda \subset \mathbb{Z}^k$ and a cost vector $l \in \mathbb{Q}^k_{>0}$, we are concerned with the family of the group problems

$$\min\{l \cdot x : x \equiv r (\mathrm{mod}\ \Lambda), x \geq 0\}, \quad r \in \mathbb{Z}^k. \tag{0.1}$$

The *lattice programming gap* $\mathrm{gap}(\Lambda, l)$ is the largest value of the minima in (0.1) as r varies over $\mathbb{Z}^k$. We show that computing the lattice programming gap is NP-hard when k is a part of input. We also obtain lower and upper bounds for $\mathrm{gap}(\Lambda, l)$ in terms of l and the determinant of Λ.

1. Introduction and statement of results

Consider the integer programming problem

$$\min\{c \cdot x : Ax = b,\ x \geq 0,\ x \text{ is integer}\}. \tag{1.1}$$

Gomory [10] defined a group relaxation of (1.1) as follows. Let B and N be the index sets of basic and nonbasic variables for an optimal basic solution to the linear programming relaxation $\min\{c \cdot x : Ax = b,\ x \geq 0\}$ of (1.1). Then the problem (1.1) can be written as

$$\min\{c_B \cdot x_B + c_N \cdot x_N : A_B x_B + A_N x_N = b, x_B,$$
$$x_N \geq 0, x_B, x_N \text{ are integer}\} \tag{1.2}$$

and a relaxation of (1.2) is obtained by removing the restriction $x_B \geq 0$:

$$\min\{c_B \cdot x_B + c_N \cdot x_N : A_B x_B + A_N x_N = b,$$
$$x_N \geq 0, x_B, x_N \text{ are integer}\}. \tag{1.3}$$

Hence (1.3) is a lower bound for (1.1) and it can be used in any branch and bound procedure.

The constraints $A_B x_B + A_N x_N = b$ in (1.3) can be written in the equivalent form $x_B = A_B^{-1} b - (A_B^{-1} A_N) x_N$. Thus, given any nonnegative integral vector x_N, the vector x_B is integer if and only if $(A_B^{-1} A_N) x_N \equiv A_B^{-1} b (\mathrm{mod}\ 1)$. Setting $c'_N = c_N - c_B A_B^{-1} A_N$, we can rewrite (1.3) as

$$\min\{c'_N \cdot x_N : (A_B^{-1} A_N) x_N \equiv A_B^{-1} b (\mathrm{mod}\ 1),$$
$$x_N \geq 0, x_N \text{ is integer}\}. \tag{1.4}$$

The program (1.4) is called *Gomory's group relaxation* for (1.1).

In this paper we fix a cost vector $c \in \mathbb{Q}^n$ and for a matrix $A \in \mathbb{Z}^{d \times n}$ of rank d and $b \in \mathrm{Sg}(A) = \{Au : u \in \mathbb{Z}^n_{\geq 0}\}$ consider the integer program

$$IP_c(A, b) = \min\{c \cdot x : Ax = b,\ x \in \mathbb{Z}^n_{\geq 0}\}.$$

For simplicity, we assume that the cone $\mathrm{cone}(A) = \{Ax : x \geq 0\}$ is pointed and that the subspace $A^{\perp} = \{x \in \mathbb{R}^n : Ax = 0\}$, the kernel of A, intersects the nonnegative orthant $\mathbb{R}^n_{\geq 0}$ only at the origin. This assumption guarantees that $IP_c(A, b)$ is bounded for all $b \in \mathrm{Sg}(A)$.

Consider the $(n-d)$-dimensional lattice $\mathcal{L}(A) = A^{\perp} \cap \mathbb{Z}^n$. The program $IP_c(A, b)$ is equivalent to the lattice program

$$\min\{c \cdot x : x \equiv u (\mathrm{mod}\ \mathcal{L}(A)),\ x \geq 0\}, \tag{1.5}$$

where u is any integer solution of the equation $Ax = b$.

A subset τ of $\{1, \ldots, n\}$ partitions $x \in \mathbb{R}^n$ as x_τ and $x_{\bar\tau}$, where x_τ consists of the entries indexed by τ and $x_{\bar\tau}$ the entries indexed by the complimentary set $\bar\tau$. Similarly, the matrix A is partitioned as A_τ and $A_{\bar\tau}$. Let τ be the set of indices of the basic variables for an optimal solution to the linear relaxation $LP_c(A, b) = \min\{c \cdot x : Ax = b, x \geq 0\}$ of the integer program $IP_c(A, b)$. Let π_τ be the projection map from $\mathbb{R}^n$ to $\mathbb{R}^{n-d}$ that forgets all coordinates indexed by τ and let $\Lambda(A) = \pi_\tau(\mathcal{L}(A))$. The lattices $\mathcal{L}(A)$ and $\Lambda(A)$ are isomorphic (see e.g. Section 2 in [23]) and Gomory's group relaxation for $IP_c(A, b)$ is equivalent to the *lattice program*

$$\min\{c'_{\bar\tau} \cdot x : x \equiv u_{\bar\tau} (\mathrm{mod}\ \Lambda(A)),\ x \geq 0\}, \tag{1.6}$$

where $c'_{\bar\tau} = c_{\bar\tau} - c_\tau A_\tau^{-1} A_{\bar\tau}$. Note that the vector $c'_{\bar\tau}$ is nonnegative. For simplicity we will consider in this paper the *generic* case, when all entries of $c'_{\bar\tau}$ are positive.

E-mail address: alievi@cardiff.ac.uk.

http://dx.doi.org/10.1016/j.orl.2015.01.008
0167-6377/

The group relaxations can be defined for various sets of variables. Wolsey [24] introduced the *extended group relaxations* obtained by dropping nonnegativity restrictions on the variables indexed by each subset of τ. Hoşten and Thomas [15] studied the set of all group relaxations obtained by dropping nonnegativity restrictions on the variables indexed by each face of a polyhedral complex associated with A and c. For further details on the classical theory of group relaxations we refer the reader to [16,1].

In this paper we will consider the group relaxations in the following general form. For a fixed cost vector $l \in \mathbb{Q}^k_{>0}$, a k-dimensional lattice $\Lambda \subset \mathbb{Z}^k$ and $r \in \mathbb{Z}^k$ we are concerned with the lattice program (also referred to as the *group problem*)

$$\min\{l \cdot x : x \equiv r(\text{mod } \Lambda), x \geq 0\}. \tag{1.7}$$

Let $m(\Lambda, l, r)$ denote the value of the minimum in (1.7). We are interested in the *lattice programming gap* $\text{gap}(\Lambda, l)$ of (1.7) defined as

$$\text{gap}(\Lambda, l) = \max_{r \in \mathbb{Z}^k} m(\Lambda, l, r). \tag{1.8}$$

The lattice programming gaps were introduced and studied for sublattices of all dimensions in $\mathbb{Z}^k$ by Hoşten and Sturmfels [14]. The algebraic and algorithmic results on the lattice programming gaps obtained in [14] have applications to the statistical theory of multidimensional contingency tables.

For fixed k the value of $\text{gap}(\Lambda, l)$ can be computed in polynomial time (see Section 3 in [14] and [7]). The first result of this paper shows that computing $\text{gap}(\Lambda, l)$ is NP-hard when k is a part of input.

Theorem 1.1. *Computing* $\text{gap}(\Lambda, l)$ *is NP-hard.*

The proof of Theorem 1.1 is based on a connection between the lattice programming gaps and the Frobenius numbers. Computing Frobenius numbers is NP-hard due to the well-known result of Ramírez Alfonsín [20].

Our next goal is to obtain the lower and upper bounds for $\text{gap}(\Lambda, l)$ in terms of the parameters of the lattice program (1.7). The bounds on the lattice programming gap provide bounds on the possible objective solutions when considering Gomory's group relaxation type problems. We show that the obtained lower bound is optimal and that the upper bound has the optimal order. The proofs are based on recent results of Marklof and Strömbergsson [19] on the diameters of circulant graphs and on the estimates of Fukshansky and Robins [9] for the Frobenius numbers.

For a given closed bounded convex set K with nonempty interior in $\mathbb{R}^k$ and a k-dimensional lattice $\Lambda \subset \mathbb{R}^k$, the *covering radius* of K with respect to Λ is defined as $\rho(K, \Lambda) = \min\{r > 0 : rK + \Lambda = \mathbb{R}^k\}$. Let X_k be the set of all k-dimensional lattices $\Lambda \subset \mathbb{R}^k$ of determinant one, let $\Delta = \{x \in \mathbb{R}^k_{\geq 0} : \sum_{i=1}^k x_i \leq 1\}$ be the standard k-dimensional simplex and let $\rho_k = \inf_{\Lambda \in X_k} \rho(\Delta, \Lambda)$. We obtain the following optimal lower bound for $\text{gap}(\Lambda, l)$.

Theorem 1.2. (i) *For any* $l \in \mathbb{Q}^k_{>0}$, $k \geq 2$, *and any* k*-dimensional lattice* $\Lambda \subset \mathbb{Z}^k$

$$\text{gap}(\Lambda, l) \geq \rho_k(\det(\Lambda) l_1 \cdots l_k)^{1/k} - \sum_{i=1}^k l_i. \tag{1.9}$$

(ii) *For any* $c \in \mathbb{Q}^{k+1}_{>0}$, $k \geq 2$, *and any* $\epsilon > 0$, *there exists a matrix* $A \in \mathbb{Z}^{1\times(k+1)}$ *such that for all* $b \in \text{Sg}(A)$ *the knapsack problem* $LP_c(A, b)$ *has a unique solution with nonbasic variables indexed by* $\sigma = \{1, \ldots, k\}$ *and for* $l = c'_\sigma$

$$\text{gap}(\Lambda(A), l) < (\rho_k + \epsilon)(\det(\Lambda(A)) l_1 \cdots l_k)^{1/k} - \sum_{i=1}^k l_i. \tag{1.10}$$

Furthermore, there exists $b' \in \text{Sg}(A)$ *such that the optimal value of* $IP_c(A, b')$ *is equal to* $\text{gap}(\Lambda(A), l) + c_{\bar\sigma} A_{\bar\sigma}^{-1} b'$.

The only known values of ρ_k are $\rho_1 = 1$ and $\rho_2 = \sqrt{3}$ (see [8]). It was proved in [2], that $\rho_k > (k!)^{1/k}$. Thus we obtain the following estimate.

Corollary 1.1. *For any* $l \in \mathbb{Q}^k_{>0}$, $k \geq 2$, *and any* k*-dimensional lattice* $\Lambda \subset \mathbb{Z}^k$

$$\text{gap}(\Lambda, l) > (k! \det(\Lambda) l_1 \cdots l_k)^{1/k} - \sum_{i=1}^k l_i. \tag{1.11}$$

For sufficiently large k the bound (1.11) is not far from being optimal. Indeed, $\rho_k \leq (k!)^{1/k}(1 + O(k^{-1} \log k))$ (cf. [6]).

Group relaxations provide the lower bounds for integer programs $IP_c(A, b)$. From this viewpoint, part (i) of Theorem 1.2 and corollary (1.11) estimate the largest possible value that such a bound can take. Part (ii) of Theorem 1.2 also shows that the obtained result is optimal in the case of knapsack problems.

Let $|\cdot|$ denote the Euclidean norm and let γ_k be the k-dimensional Hermite constant (see i.e. Section IX.7 in [5]). We give the following upper bound for $\text{gap}(\Lambda, l)$ (and hence for the minimum in (1.6)).

Theorem 1.3. *For any* $l \in \mathbb{Q}^k_{>0}$, $k \geq 2$, *and any* k*-dimensional lattice* $\Lambda \subset \mathbb{Z}^k$

$$\text{gap}(\Lambda, l) \leq \frac{k\gamma_k^{k/2} \det(\Lambda)\left(\sum_{i=1}^k l_i + |l|\right)}{2} - \sum_{i=1}^k l_i. \tag{1.12}$$

The known exact values of γ_k^k are 1, 4/3, 2, 4, 8, 64/3, 64, 256 (Sloan's sequence A007361 in [22]). By a result of Blichfeldt (see, e.g. [13]) $\gamma_k \leq 2\left(\frac{k+2}{\sigma_k}\right)^{2/k}$, where σ_k is the volume of the unit k-ball; thus $\gamma_k = O(k)$. The precision of the bound (1.12) depends on the estimates for the covering radius of a simplex, associated with the cost vector l, with respect to the lattice Λ. It follows from results in [3, Section 6] that the order $\text{gap}(\Lambda, l) = O_{k,l}(\det(\Lambda))$, where the constant depends on k and l, cannot be improved.

A widely used approach (see e.g. [4]) is to consider a group relaxation induced by a single row i: $\sum_{j\in N} \hat{a}_{ij} x_j \equiv \hat{b}_i (\text{mod } 1)$ of the matrix constraint in (1.4). Here we may assume that all $\hat{a}_{ij}$ and $\hat{b}_i$ are rational numbers from $[0, 1)$ with common denominator $D = |\det(B)|$. Thus, multiplying by D, we get the constraint $\sum_{j\in N}(D\hat{a}_{ij})x_j \equiv D\hat{b}_i(\text{mod } D)$. Set $k = |N|, A = (D\hat{a}_{i1}, \ldots, D\hat{a}_{ik}, D) \in \mathbb{Z}^{1\times(k+1)}$ and $\Lambda = \pi_{\{k+1\}}(\mathcal{L}(A))$. We may assume that $l = c'_{\bar\tau} \in \mathbb{Q}^k_{>0}$, where τ is the set of indices of basic variables. Then for any integer solution $r \in \mathbb{Z}^k$ of $r \cdot \pi_{\{k+1\}}(A) \equiv D\hat{b}_i(\text{mod } D)$ the group relaxation induced by the row i can be written in the form (1.7). Thus all bounds derived in this paper can be applied to the group relaxation induced by a selected row of (1.4). Note that in this special case the lattice programming gap $\text{gap}(\Lambda, l)$ can be associated with the diameter of a directed circulant graph (see [19] for details). Furthermore, the results of [19] show that the lower bound (1.9) is a good predictor for the value of $\text{gap}(\Lambda, l)$ for a 'typical' Λ.

2. $\text{gap}(\Lambda, l)$ and diameters of quotient lattice graphs

Assume for the rest of the paper $k \geq 2$. Following notation from [19], let $LG_k^+ = (\mathbb{Z}^k, E)$ be the standard directed lattice graph with vertex set $\mathbb{Z}^k$. The edge set E consists of all directed edges $(x, x + e_j)$, where $x \in \mathbb{Z}^k$ and $e_1, \ldots, e_k$ are the standard basis vectors. Let Λ be a k-dimensional sublattice of $\mathbb{Z}^k$. We define the quotient lattice graph LG_k^+/Λ as the digraph with vertex set $\mathbb{Z}^k/\Lambda$ and the edge set $\{(x+\Lambda, x+e_j+\Lambda) : x \in \mathbb{Z}^k, j = 1, \ldots, k\}$. Given cost vector $l \in \mathbb{Q}^k_{>0}$, we define the distance from vertex $x + \Lambda$ to