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# A polynomial time algorithm for the single-item lot sizing problem with capacities, minimum order quantities and dynamic time windows

Bertrand Hellion [a,*], Fabien Mangione [b,c], Bernard Penz [b,c]

[a] *AZAP, the supply chain optimiser, 75 rue de Courcelles, 75018 Paris, France*
[b] *G-SCOP, Université Grenoble Alpes, G-SCOP, F-38000 Grenoble, France*
[c] *CNRS, G-SCOP, F-38000 Grenoble, France*

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ABSTRACT

This paper deals with the single-item capacitated lot sizing problem with concave production and storage costs, considering minimum order quantity and dynamic time windows. The frequency constraints on the production lots are modeled by dynamic time windows. Between two consecutive production lots, there are at least $Q$ periods and at most $R$ periods. This paper presents an optimal algorithm in $O((T-Q)^2\frac{(R-Q)T^4}{Q^3})$, **which is bounded by** $O(T^7)$.



## 1. Introduction

This paper deals with a generalization of the single-item capacitated lot sizing problem (CLSP) with fixed capacity. The CLSP consists in satisfying a demand at each time period $t$ over a planning horizon $T$. The demand is satisfied from stock or by production. Costs incur for each item produced and also when an item is stored between two consecutive periods. A fixed maximum production capacity ($U$) must be respected. The problem considered in this paper contains a minimum order quantity constraint (MOQ). This constraint imposes that if an item is produced at a given period, the quantity must be greater than or equal to a minimum level $L$. The $U$ and $L$ values are constant over the $T$ periods. This problem also includes dynamic time windows (DTW). Between two consecutive production lots, there are at least $Q$ periods and at most $R$ periods.

In a long term partnership, a stability contract can be established between a supplier and a retailer with the objective of stabilizing their deliveries. This can lead to a number of advantages for the supplier as detailed in Hellion et al. [11]. Hellion et al. [11] show that the implementation of a stability contract increases the retailer's storage costs, which can be equilibrated by a price discount on the behalf of the supplier. *The overall estimated profit for the whole supply chain is shown to be proportional to the hardness of the stability contracts used.*If a stability contract is established between the supplier and the retailer (which consists in finding the best values for $U$, $L$, $Q$ and $R$), tactical planning must take these values into account as constraints in the optimization of supply. In particular, the retailer can optimize its planning by solving a single-item lot sizing problem with capacities ($U$), minimum order quantities (MOQ or $L$) and dynamic time windows ($Q$, $R$). This problem is noted as CLSP–MOQ–DTW in the following.

The single-item capacitated lot sizing problem is known to be *NP*-Hard [3]. However, some cases are polynomially solvable. This is the case when the capacity is fixed over the $T$ periods. Florian and Klein [8] considered a case where production and holding cost functions are concave. They proposed an exact method with a time complexity in $O(T^4)$. Later van Hoesel and Wagelmans [20] improved the complexity of the algorithm in $O(T^3)$ when the holding costs are linear. A complete survey on the single-item lot sizing problem can be found in [4].

Recently minimum order quantity (MOQ) constraints have been developed. These constraints deal with the production level that must be at least the MOQ if the production is to be started. The

* Corresponding author.
*E-mail addresses:* bertrand.hellion@gmail.com, bhellion@azap.com (B. Hellion).

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CLSP–MOQ has been shown relevant in many industrial contexts, for example Lee [13] has studied an industrial problem where a manufacturer imposes a minimum order quantity to its supplier. Furthermore, Porras and Dekker [17] have worked on an industrial case where the producer imposes minimum order quantities (MOQ) to produce the items. Zhou et al. [22] have analyzed a class of simple heuristic policies to control stochastic inventory systems with MOQ constraints. They also developed insights into the impact of MOQ constraints on repeatedly ordered items to fit in an industrial context. Okhrin and Richter [16] develop the first exact polynomial time algorithm for this problem. They solved a special case of the problem in which the unit production cost is constant over the whole horizon and then can be discarded. Furthermore, they assumed that the holding costs are also constant over the $T$ periods, with these restrictions they derived a polynomial time algorithm in $O(T^3)$. Li et al. [15] studied the single item lot sizing problem with lower bounds and described a polynomial algorithm in $O(T^7)$ to solve the special case with concave production and storage cost function. Later Hellion et al. [9,10] developed an optimal $O(T^6)$ polynomial time algorithm to solve the CLSP–MOQ with concave costs functions, improving Li et al. [15] algorithm. Hellion et al. [9,10] also provide a computational experiment to underline the practical complexity of their algorithm.

The production capacity and MOQ constraints were originally motivated by industrial needs. Considering a retailer ordering from a single supplier, these constraints additionally give the supplier a way to forecast future orders. However, these constraints only affect the quantity of the orders, and both the supplier and the retailer lack temporal information. To ensure a long-term partnership, actors must guarantee a certain amount of supplied components and regular orders. The time interval between two orders must be in a given time window [11].

In the existing literature, time windows have been introduced with several definitions. Dauzère-Pérès et al. [7] introduce production time windows. Each demand $d_i$ (with $i = 1, 2 \dots n$) is defined by a time window $[E_i, L_i]$ during which a replenishment for the demand must occur. At the end of the time window the produced quantity is dispatched to satisfied the demand. Time windows can be used to model perishability (whereby products cannot be stored indefinitely in a warehouse). The initial model is extended to the capacitated multi-item problem by Brahimi et al. [4]. Recently, Absi et al. [1] studied two production time window problems, considering lost sales or backlogs, using dynamic programming. Hwang [12] proposed an $O(T^5)$ algorithm for production time windows and concave production costs. van den Heuvel [19] showed that the formulations with production time windows are equivalent to other models: lot sizing with manufacturing options, lot sizing with cumulative capacities and lot sizing with inventory bounds.

On the other hand, Lee et al. [14] present delivery time windows (also called grace period). In this model, demand is not defined with a single period ($d_t$) but by an interval $d_{t_1<t_2}$ with $t_1, t_2 \in T^2$. This demand must be satisfied in any period between $t_1$ and $t_2$. Later, Akbalik and Penz [2] used a similar definition to compare just-in-time and time windows policies. These two time window definitions were studied by Wolsey [21], he proposed valid inequalities and convex hulls.

However, these time window definitions above do not guarantee regular orders. Hellion et al. [11] recently presented a new time windows definition in which actors have to agree on a minimum and a maximum number of periods between two orders. Since an order is dependent on the period where the last order occurred, these time windows are dynamic (called DTW as already defined).

In this paper, we extend Hellion et al.'s algorithm [9,10] to the problem with dynamic time windows. The paper is organized as follows: Section 2 describes the problem and introduces the notations. Section 3 presents the necessary definitions and properties to give a polynomial algorithm, and the algorithm. Finally, concluding remarks and perspectives are given in Section 4.

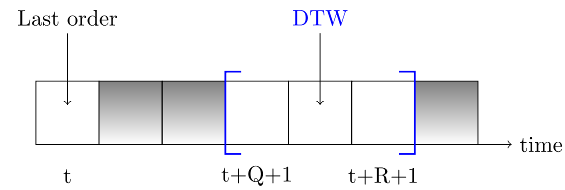


**Fig. 1.** An example of dynamic time window with $Q = 2$ and $R = 4$.

## 2. Problem description and notations

### 2.1. Description

The single item lot sizing problem consists of satisfying the demands $d_t$ of a product at each period $t$ over $T$ consecutive periods. A demand $d_t \in \mathbb{Z}^+$ may be satisfied by the production of an item at period $t$ ($X_t$) and/or from inventory ($I$) available at the end of the period $t - 1$ ($I_{t-1}$). Backlogs are not allowed. The inventory level at the end of a period $t$ is denoted $I_t$. It is assumed without loss of generality that there is no inventory at the beginning of the first period. The problem is to determine the amount $X_t$ to be produced at each period, satisfying the demands and minimizing the total cost.

A constant capacity $U$ constrains the production at each period. A minimum order quantity (MOQ) $L$ also constrains the production level. Each subsequent production level is also constrained by a dynamic time window (DTW). There are at least $Q$ and at most $R$ periods between two consecutive production lots.

Fig. 1 illustrates the dynamic time window for $Q = 2$ and $R = 4$. In the example, a lot is produced in period $t$. Since $Q = 2$, the following lot cannot be produced at either at period $t+1$ or at $t+2$. Since $R = 4$, at least a lot must be produced in the next five periods. Thereafter, the next lot must be produced between periods $t + 3$ ($t + Q + 1$) and $t + 5$ ($t + R + 1$) included. Consequently, in each interval of length 3 at most one lot must be produced. Furthermore, in each interval of length 5, at least one lot must be produced. Note that if $Q = R$ one lot must be produced every $Q + 1$ periods. If $Q = R = 0$, one lot must be produced every period.

The production cost is a concave function of the quantity produced $p_t(X_t)$ and the inventory cost is a concave function of the inventory level $h_t(I_t)$. Note that concave cost functions may include set-up costs.

### 2.2. Mathematical formulation

The mathematical formulation is now presented. The decision variables are given as follows:

- $X_t$: quantity of products ordered at period $t$.
- $Y_t = \begin{cases} 1 & \text{if an order is placed at period } t. \\ 0 & \text{otherwise.} \end{cases}$
- $I_t$ = inventory level at the end of a period $t$.

The mathematical formulation of the CLSP–MOQ–DTW is then:

$$\text{Min} \sum_{t=1}^{T} p_t(X_t) + \sum_{t=1}^{T} h_t(I_t) \qquad (1)$$

$$X_t + I_{t-1} - I_t = d_t \quad \forall t \in T \qquad (2)$$

$$LY_t \le X_t \le UY_t \quad \forall t \in T \qquad (3)$$

$$\sum_{t'=t}^{t+R} Y_{t'} \ge 1 \quad \forall t \in \{1, \dots, T - R\}, \qquad (4)$$

$$\sum_{t'=t}^{t+Q} Y_{t'} \le 1 \quad \forall t \in \{1, \dots, T - Q\} \qquad (5)$$

$$X_t, I_t \in \mathbb{R} \quad \forall t \in T \qquad (6)$$

$$Y_t \in \{0, 1\} \quad \forall t \in T. \qquad (7)$$