Operations Research Letters 42 (2014) 505–508

Contents lists available at ScienceDirect

Operations Research Letters

journal homepage: www.elsevier.com/locate/orl

On characterization of the core of lane covering games via dual solutions

Behzad Hezarkhani *, Marco Slikker, Tom Van Woensel

School of Industrial Engineering, Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, The Netherlands

ARTICLE INFO

Article history:
Received 7 November 2013
Received in revised form
4 September 2014
Accepted 4 September 2014
Available online 16 September 2014

Keywords:
Logistics
Lane covering problem
Cooperative game theory
Core

ABSTRACT

The lane covering game (LCG) is a cooperative game where players cooperate to reduce the cost of cycles that cover their required lanes on a network. We discuss the possibilities/impossibilities of a complete characterization of the core via dual solutions in LCGs played among a collection of shippers, each with a number of service requirements along some lanes, and show that such a complete characterization is possible if each shipper has at most one service requirement.

1. Introduction

In any cooperative situation, the division of joint costs is a critical issue. The *core* of a cooperative game contains allocations that provide players with sufficient incentives to remain in the grand coalition. In general, finding an allocation in the core and testing the core-membership of a given allocation are computationally difficult problems as they involve dealing with a number of inequalities which grow exponentially in the number of players. In linear production games [4], every solution to the corresponding dual linear program yields an allocation in the core thus core allocations can be found in polynomial time [6]. Although testing the membership of a given allocation to the core for this class of games is generally co-NP-complete [2], in some cases, e.g. flow games on simple networks [3], dual solutions obtain *all* allocations in the core. This note addresses the possibilities/impossibilities of a complete characterization of the core via dual solutions in *lane covering games*.

The lane covering game (LCG), introduced by Özener and Ergun [5], can be represented as an instance of linear production games where players cooperate to reduce the cost of cycles that cover their required lanes. Consider a collection of locations and the network of roads in between. There are several shippers who provide truckload deliveries between pairs of locations. After fulfilling its planned deliveries across required lanes, every shipper must return to its starting location (repositioning). By collaboration, shippers can reduce the total repositioning cost needed for fulfilling their consolidated deliveries. Özener and Ergun [5] show that if each required lane is considered to be a single player, the dual solutions completely characterize the core of corresponding game. We extend and complete this result by allowing *shippers* to be the actual players. Each shipper might have several service requirements (across one or multiple lanes). We specify the situations in which the core can or cannot be completely characterized by dual solutions. The main contribution of this note is to prove that a complete characterization of the core via duals is possible if every shipper has at most one service requirement. We also provide examples of LCGs where such a complete characterization fails.

2. Cooperative cost games and core

Consider a set P of distinct players. A cooperative cost game is a pair (P, z) with $z : 2^P \rightarrow \mathbb{R}$ being the coalition function that assigns to every coalition $S \subseteq P$ the cost $z(S)$. A critical problem in cooperative cost games is finding appropriate allocations.

An allocation $\beta = (\beta^k)_{k\in P}$ is a vector containing a real number for every player in P. The *core* of the game (P, z) is the set of all allocations β such that

$$\sum_{k\in P} \beta^k = z(P) \quad (1)$$

and

$$\sum_{k\in S} \beta^k \leq z(S), \quad \forall S \subseteq P. \quad (2)$$

* Corresponding author.
E-mail addresses: b.hezarkhani@tue.nl (B. Hezarkhani), m.slikker@tue.nl (M. Slikker), t.v.woensel@tue.nl (T. Van Woensel).

http://dx.doi.org/10.1016/j.orl.2014.09.002
0167-6377/

Allocations that satisfy the equality in (1) are called *efficient*. The efficiency condition requires that allocations divide the entire cost of the grand coalition among the players. Allocations that satisfy the collection of inequalities in (2) are called *stable*. The stability condition mandates that the total allocation to any group of players does not exceed the cost of their corresponding coalition. The core contains all efficient and stable allocations.

3. Lane covering situations and games

Consider a geographical region including a number of locations and the interconnecting roads among them. A number of independent shippers operate in this region. In a given planning period, each shipper has a certain number of orders for providing service across pairs of locations. An example of such service is the transportation of full-truck cargo that, in its simplest form, requires a shipper to utilize a vehicle in order to pickup cargo at one location and drop it off at another location. The vehicles utilized by the shippers are stationed in designated locations (depots) and must return to their designated locations after completing their scheduled deliveries. Therefore, required services must be fulfilled via cycles. In order to construct service cycles, the vehicles may have to travel in between locations which do not require service (repositioning movements).

To formalize the above situation, consider the complete directed graph $G = (N, A)$ where N is a finite set of nodes representing the spatial locations and $A = \{ij | i, j \in N, i \neq j\}$ is the set of ordered lanes representing the network of roads. The service cost vector $c = (c_{ij})_{ij \in A}$ gives the non-negative costs of providing service across the lanes. Traversing lane ij without providing service would cost θc_{ij} with $0 \leq \theta \leq 1$. A set of shippers (players) P operate on G. A given player $k \in P$ has an individual *requirement vector* $r^k = (r^k_{ij})_{ij \in A}$ where $r^k_{ij} \in \mathbb{N} \cup \{0\}$ is the number of service requirements of k along the lane ij. For example, $r^k_{ij} = 2$ implies that player k must provide service across lane ij twice. A player k is called a *simple shipper* if $\sum_{ij \in A} r^k_{ij} = 1$. That is, a simple shipper has a single service requirement. We define a *lane covering situation* as the tuple

$$\Gamma = \left(G, c, \theta, P, (r^k)_{k \in P}\right).$$

Cooperation among the shippers enables them to take advantage of the potential synergies in their requirements and minimize their joint repositioning costs. However, the shippers have to decide on the allocation of overall costs as well. To address this issue, we construct and analyze cooperative cost games associated with lane covering situations.

The *lane covering game* (LCG) associated with the situation Γ is a cooperative cost game (P, z^Γ) where $z^\Gamma(S)$ is the minimum cost of covering the service requirements of coalition $S \subseteq P$ via cycles. The requirements of coalition S are the sum of the requirements of its individual members, i.e. $r^S = \sum_{k \in S} r^k$. For any $S \subseteq P$, $z^\Gamma(S)$ can be obtained through an integer linear program. Let x_{ij} and w_{ij} denote the number of times that lane ij is traversed with and without service respectively. We have

$$\textbf{Model 1}: z^\Gamma(S) = \min \sum_{ij \in A} c_{ij} x_{ij} + \theta c_{ij} w_{ij} \quad (3)$$

$$\text{s.t.} \sum_{j \in N \setminus \{i\}} x_{ij} - x_{ji} + w_{ij} - w_{ji} = 0 \quad \forall i \in N \quad (4)$$

$$x_{ij} \geq r^S_{ij} \quad \forall ij \in A \quad (5)$$

$$x_{ij}, w_{ij} \in \mathbb{N} \cup \{0\} \quad \forall ij \in A. \quad (6)$$

The flow conservation constraints in (4) guarantee that requirements are fulfilled in cycles and constraints in (5) ensure that all requirements of coalition S are met. We denote an optimal solution for the above problem with $(x^S_{ij}; w^S_{ij})_{ij \in A}$.

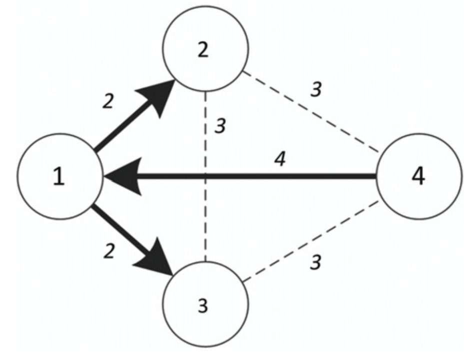

Fig. 1. The network in Example 1.

Model 1 corresponds to a minimum-cost circulation problem with its constraints forming a totally unimodular matrix [6]. Thus with integer requirement vectors, the linear relaxation of Model 1 does not affect the optimal solution.

The dual associated with the linear relaxation of Model 1 for the grand coalition P is

$$\textbf{Model 2}: d^\Gamma = \max \sum_{ij \in L} r^P_{ij} I_{ij} \quad (7)$$

$$\text{s.t. } I_{ij} + y_i - y_j \leq c_{ij} \quad \forall ij \in A \quad (8)$$

$$y_i - y_j \leq \theta c_{ij} \quad \forall ij \in A \quad (9)$$

$$I_{ij} \geq 0 \quad \forall ij \in A \quad (10)$$

where $L = \{ij | r_{ij} > 0\}$ is the *set of required lanes* in situation Γ. Let $I^\Gamma = (I^\Gamma_{ij})_{ij \in A}$ be an optimal solution to d^Γ. For a required lane $ij \in L$, I^Γ_{ij} gives the shadow price that determines the amount of decrease in $z^\Gamma(P)$ resulting from reducing r^P_{ij} by one. We denote the set of all solutions to d^Γ with $\boldsymbol{I}^\Gamma$.

4. Dual allocations

Owen [4] introduces the class of *linear production games* and shows that an allocation in the core of these games can be obtained from a solution to the dual problem. As discussed in Özener and Ergun [5], the game (P, z^Γ) with $z^\Gamma(S)$ defined by the LP-relaxation of Model 1 for every $S \subseteq P$ is an instance of the class of linear production games. Accordingly, an allocation in the core of (P, z^Γ) can be obtained from a dual solution in the following manner:

$$\beta^k = \sum_{ij \in L} r^k_{ij} I^\Gamma_{ij}, \quad \forall k \in P. \quad (11)$$

Thus, in LCGs every dual solution obtains an allocation in the core. The question concerning a complete characterization of core via dual solutions addresses the reverse of the latter, i.e. does every core allocation correspond to a dual solution?

5. LCGs with general shippers

In this section we show that a complete characterization of the core via duals is not possible if some players have multiple service requirements (general shippers). The following example shows that this is the case even if every lane requires service at most once.

Example 1. Consider the lane covering situation associated with the graph in Fig. 1. The service costs across lanes with opposite directions are symmetric and are given in the figure. We let $\theta = 1$. Consider two players $P = \{A, B\}$ with player A requiring service on lanes 12 and 13, and player B requiring service along the lane 41.