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A new Pareto set generating method for multi-criteria optimization problems

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ABSTRACT

This paper proposes a new classical method to capture the complete Pareto set of a multi-criteria optimization problem (MOP) even without having any prior information about the location of *Pareto surface*. The solutions obtained through the proposed method are globally Pareto optimal. Moreover, each and every global Pareto optimal point is within the attainable range. This paper also suggests a procedure to ensure the *proper Pareto optimality* of the outcomes if slight modifications are allowed in the constraint set of the MOP under consideration. Among the set of all outcomes, the proposed method can effectively detect the regions of unbounded trade-offs between the criteria, if they exist.

1. Introduction

The engineering design problem, one of the best examples of the realistic decision problem, mostly involves simultaneous optimization of multiple criteria with several equality and inequality constraints. These criteria often do not agree with each other. Hence, there exists a set of optimal solutions that are actually compromise solutions of the MOP. Under certain conditions, a set of compromise solutions leads to the foundation of the concept of *Pareto optimality* [16]. A *Pareto optimal solution* is a feasible point in the solution space where any improvement of one criterion can take place only through the worsening of at least one criterion, other than the aforesaid one. Some results of the characterization of Pareto optimal points can be obtained in [20]. All the Pareto optimal solutions play a very important role in MOPs when it comes to the analysis of the trade-off among the conflicting criteria. Over the decades, many classical methods [11], like, weighted sum [8,9,21], ϵ-constraint [6,14], compromise programming [22], physical programming [12], normal boundary intersection [4], normal constraint [13], direct search domain [7], etc. have been used to capture the Pareto set of an MOP. A quasi-Newton-type algorithm for solving MOPs is developed in [17]. An excellent literature survey on Pareto set generating classical techniques can be found in [7,11,14]. Erfani and Utyuzhnikov [7] have mentioned that the existing classical methods either cannot capture the complete Pareto set or require some prior information about the location of the Pareto surface. Thus, in this paper, an attempt has been made to develop a classical method that does not have any of these deficiencies. The proposed research work is organized as follows.

In the next section, preliminaries and notations on MOPs, which are used throughout the paper, are given. In Section 3, we propose a classical method to generate the complete Pareto set. Related results and discussions on the developed method are demonstrated in Section 4. Algorithmic implementation of the proposed method is given in Section 5. In Section 6, a brief comparison between the proposed method and the already existing methods that are similar to the proposed one is presented. Lastly, Section 7 contains suggestions for future research and a conclusion to the study.

2. Preliminaries

In mathematical notation, MOPs are defined in the following way

$$\min_{x\in\mathcal{X}} f(x) = \big(f_1(x), f_2(x), \ldots, f_k(x)\big)^t, \quad k \geq 2, \tag{1}$$

where $\mathcal{X} = \{x \in \mathbb{R}^n : g(x) \leq 0,\ h(x) = 0,\ a \leq x \leq b\}$ is the feasible set, $g : \mathbb{R}^n \to \mathbb{R}^r$ and $h : \mathbb{R}^n \to \mathbb{R}^s$ are vector valued functions and the constant vectors $a, b \in (\mathbb{R} \cup \{-\infty\})^n$ are respectively the lower and the upper bound of the decision vector $x = (x_1, x_2, \ldots, x_n)^t$.

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We denote the image of the decision feasible set $\mathcal{X}$ under the vector mapping f by $\mathcal{Y} := f(\mathcal{X})$. Therefore, $\mathcal{Y}$ is the feasible set in the criterion space. If for each individual $i \in \{1, 2, \ldots, k\}$, the global minimum of the function f_i is x_i^*, then the point $y_i^* := f(x_i^*) = f_i^*$ is said to be an *anchor point*. The point $f^* = (f_1^*, f_2^*, \ldots, f_k^*)^t$ is called *utopia point* [11]. Commonly the utopia point f^* is not attainable by f. Thus the notion of Pareto optimality is being introduced as follows. Definitions of weak Pareto optimality and proper Pareto optimality are also given subsequently.

The definition of Pareto optimality depends on a dominance structure or on a componentwise order in the space of $\mathbb{R}^k$. To represent dominance structure on $\mathbb{R}^k$, the following subsets are used. The non-negative orthant of $\mathbb{R}^k$ is represented by $\mathbb{R}^k_{\geqq} := \{y \in \mathbb{R}^k : y \geqq 0\}$. The notation $y \geqq 0$ implies $y_i \geq 0$ for each $i = 1, 2, \ldots, k$. The set $\mathbb{R}^k_{\geq}$ is defined by $\{y \in \mathbb{R}^k : y \geq 0\}$ where $y \geq 0$ means $y \geqq 0$ but $y \neq 0$. The notation $\mathbb{R}^k_{>} := \{y \in \mathbb{R}^k : y > 0\}$ indicates the positive orthant of $\mathbb{R}^k$. Here, $y > 0$ stands for $y_i > 0$ for each $i = 1, 2, \ldots, k$. The relations '$\leqq$', '$\leq$' and '$<$' are similarly defined. For two feasible points $\hat{x}$ and $\bar{x}$ in $\mathcal{X}$, the vector $f(\hat{x})$ is said to dominate $f(\bar{x})$ if $f(\hat{x}) \leq f(\bar{x})$.

Definition 2.1 (*Pareto Optimality [6]*)**.** A feasible solution—$\hat{x} \in \mathcal{X}$—is called an efficient point or a Pareto optimal point if there is no other $x \in \mathcal{X}$ such that $f(x) \leq f(\hat{x})$. If $\hat{x}$ is an efficient point, the corresponding point, $f(\hat{x})$, is said to be a non-dominated point. The set of all efficient points is denoted by $\mathcal{X}_E$. The set of all non-dominated points is represented by $\mathcal{Y}_N$.

Definition 2.2 (*Weak Pareto Optimality [6]*)**.** A feasible solution—$\hat{x} \in \mathcal{X}$—is called a weak efficient or weak Pareto optimal point if there is no $x \in \mathcal{X}$ such that $f(x) < f(\hat{x})$. The point—$\hat{y} = f(\hat{x})$—is then said to be a weak non-dominated point. The set of all weak efficient points is denoted by $\mathcal{X}_{wE}$. The collection of all non-dominated points is represented as $\mathcal{Y}_{wN}$.

Definition 2.3 (*Proper Pareto Optimality [6]*)**.** A Pareto optimal point $\hat{x}$ is said to be a properly Pareto optimal point if there exists a constant $M > 0$ such that corresponding to any $i \in \{1, 2, \ldots, k\}$ and $x \in \mathcal{X}$ satisfying $f_i(x) < f_i(\hat{x})$, we can find an index j such that $f_j(\hat{x}) < f_j(x)$ and $\frac{f_i(\hat{x})-f_i(x)}{f_j(x)-f_j(\hat{x})} \leq M$. If $\hat{x}$ is a properly efficient point, the corresponding point $f(\hat{x})$ is called a properly non-dominated point.

The point $y^N = (y_1^N, y_2^N, \ldots, y_k^N)^t$ given by $y_i^N := \max_{x \in \mathcal{X}_E} f_i(x) = \max_{y \in \mathcal{Y}_N} y_i$ is called the *nadir point* of the multi-criteria optimization problem.

It can easily be shown that a feasible point $\hat{x} \in \mathcal{X}$ belongs to $\mathcal{X}_E$ if and only if $(f(\hat{x}) - \mathbb{R}^k_{\geq}) \bigcap f(\mathcal{X}) = \{f(\hat{x})\}$. Similarly, a feasible point $\hat{x} \in \mathcal{X}$ belongs to $\mathcal{X}_{wE}$ if and only if $(f(\hat{x}) - \mathbb{R}^k_{>}) \bigcap f(\mathcal{X}) = \emptyset$. In a more general sense, if the objective space $\mathbb{R}^k$ is ordered by a pointed convex cone, say D, a feasible point $\hat{x} \in \mathcal{X}$ is said to be efficient with respect to D if $(f(\hat{x}) - D) \bigcap f(\mathcal{X}) = \{f(\hat{x})\}$. Analogously, a point $\hat{x} \in \mathcal{X}$ is weak efficient with respect to D if $(f(\hat{x}) - int(D)) \bigcap f(\mathcal{X}) = \emptyset$. If D is taken as $\mathbb{R}^k_{\epsilon} := \{y \in \mathbb{R}^k : dist(y, \mathbb{R}^k_{>}) \leq \epsilon \|y\|\}$, then a (weak) Pareto optimal point with respect to D is said to be a (weak) ϵ-Pareto optimal point. Since, at any ϵ-Pareto optimal point, trade-off between any two criteria are bounded by ϵ and $1/\epsilon$, the ϵ-Pareto optimal points are always properly Pareto optimal.

In this paper, we assume the following conditions on the MOP (1).

- *Assumptions on the MOP* (1):
  (i) *Each objective function $f_j, j = 1, 2, \ldots, k$, has minimum value 'zero' on the decision feasible set $\mathcal{X}$.*
  (ii) *The utopia point $f^* = (f_1^*, f_2^*, \ldots, f_k^*)^t$ is not a feasible point.*
  (iii) *The decision feasible region and the criterion feasible region are compact.*

Under these assumptions, a new classical technique to capture the complete non-dominated set (Pareto set) of the MOP (1) is proposed in the following section.

3. A new method

The presented technique rests on the following three noteworthy observations on Pareto optimality—

- a point $\hat{x} \in \mathcal{X}$ is a Pareto optimal point if and only if $f(\mathcal{X}) \cap \left(f(\hat{x}) - \mathbb{R}^k_{\geqq}\right) = \{f(\hat{x})\}$,
- a point $\hat{x} \in \mathcal{X}$ is a weak Pareto optimal if and only if $f(\mathcal{X}) \cap \left(f(\hat{x}) - \mathbb{R}^k_{>}\right) = \emptyset$ and
- the sets of all non-dominated points and weak non-dominated points must be subsets of the boundary of the criterion feasible region, $bd(\mathcal{Y})$.

Geometrically, the first observation signifies that—if the intersection between 'the criterion feasible region' and 'the translated non-positive orthant $-\mathbb{R}^k_{\geqq}$, whose vertex is being shifted from origin to the point $f(\hat{x})$', is only the single point $f(\hat{x})$, then $\hat{x}$ is a Pareto optimal solution. Thus, to capture a Pareto optimal point, we may translate the cone $-\mathbb{R}^k_{\geqq}$ along a particular direction $\hat{\beta} \in \mathbb{R}^k_{\geq}$ until this cone does not touch the criterion feasible region. The translation of the cone $-\mathbb{R}^k_{\geqq}$ along $\hat{\beta} \in \mathbb{R}^k_{\geq}$ must be done in such a way that the vertex of the cone is retained on the line of vectors $z\hat{\beta}, z \in \mathbb{R}$. Now, if the cone $-\mathbb{R}^k_{\geqq}$ is being translated along $\hat{\beta} \in \mathbb{R}^k_{\geq}$, i.e., if we move the cone $z\hat{\beta} - \mathbb{R}^k_{\geqq}$ with $z > 0$, then it can touch the boundary of the criterion feasible region $\mathcal{Y}$ in two possible ways, as mentioned below.

(i) *The vertex of the cone touches first.* In this case, the point where the cone touches the criterion feasible region is certainly a globally non-dominated point.
(ii) *One (or more) boundary plane(s) of the cone touches (touch) first.* In this case, two possibilities may arise: the contact portion is either a single point or a set of points. In the first subcase, the contact point is a non-dominated point. In the second subcase, it can be easily perceived that all the points except the extreme points of the contact portion are weak non-dominated points.

We illustrate how the above-mentioned contact area of the cone $z\hat{\beta} - \mathbb{R}^k_{\geqq}$ and the boundary set $bd(\mathcal{Y})$, for a particular direction $\hat{\beta} \in \mathbb{R}^k_{>}$, can be found. To demonstrate the scenario, let us consider a graphical perspective of a bi-criteria optimization problem. Fig. 1 portrays a generic bi-criteria feasible region $\mathcal{Y} = f(\mathcal{X})$ and the cone $z\hat{\beta} - \mathbb{R}^2_{\geqq}$ for a specific value of $z = \overline{OA}$. Let us now consider the set $\left\{y : z\hat{\beta} \geqq f(x), y = f(x), x \in \mathcal{X}\right\}, z \in \mathbb{R}$. For each specific value of $z \in \mathbb{R}$, this set represents the intersecting region of the cone $(z\hat{\beta} - \mathbb{R}^2_{\geqq})$ and the criterion feasible region $f(\mathcal{X})$. Now, for a generic value of $z \in \mathbb{R}$, let us try to minimize the intersecting region between $(z\hat{\beta} - \mathbb{R}^2_{\geqq})$ and $f(\mathcal{X})$ by translating the cone $(z\hat{\beta} - \mathbb{R}^2_{\geqq})$ along $\hat{\beta}$ such a way that the cone does not leave the set $f(\mathcal{X})$. In the optimum situation, if the intersection $(z\hat{\beta} - \mathbb{R}^2_{\geqq}) \bigcap f(\mathcal{X})$ contains only one point, then that singleton point is surely a non-dominated point. We note that reduction to the smallest possible amount of the intersecting region $(z\hat{\beta} - \mathbb{R}^2_{\geqq}) \bigcap f(\mathcal{X})$ eventually involves minimization of the value of z satisfying the constraints $z\hat{\beta} \geqq f(x), x \in \mathcal{X}$. It is worth noticing that the above discussion does not depend on the number of criteria or objective functions.