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Decomposition theorems for linear programs

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ABSTRACT

Given a linear program (LP) with m constraints and n lower and upper bounded variables, any solution $\mathbf{x}^0$ to LP can be represented as a nonnegative combination of at most $m+n$ so-called weighted paths and weighted cycles, among which at most n weighted cycles. This fundamental decomposition theorem leads us to derive, on the residual problem $LP(\mathbf{x}^0)$, two alternative optimality conditions for linear programming, and eventually, a class of primal algorithms that rely on an Augmenting Weighted Cycle Theorem.

1. Introduction

Network flow problems can be formulated either by defining flows on arcs or, equivalently, flows on paths and cycles, see Ahuja et al. [1]. A feasible solution established in terms of path and cycle flow determines arc flows uniquely. The converse result, that is the *existence* of a decomposition as a path and cycle flow equivalent to a feasible arc-flow solution $\mathbf{x}^0$, is also shown to be true by the *Flow Decomposition Theorem*, although the decomposition might not be unique. This result can be refined for circulation problems, establishing that a feasible circulation can be represented along cycles only. Originally developed by Ford and Fulkerson [4] for the maximum flow problem, the flow decomposition theory intervenes in various situations, notably on the residual network. It is used to prove, among many other results, the *Augmenting Cycle Theorem* and the *Negative Cycle Optimality Theorem*. The first allows to build one solution from another by a sequence of cycles. The second states that arc-flow solution $\mathbf{x}^0$ is optimal if and only if the residual network contains no negative cost cycle therefore providing optimality characterization for network flow problems. The *Flow Decomposition Theorem* is a fundamental theorem as it is an essential tool in the complexity analysis of several strongly polynomial algorithms such as the *minimum mean cycle-canceling* algorithm, see Goldberg and Tarjan [6], Radzik and Goldberg [8], and Gauthier et al. [5] for an improved complexity result. This paper generalizes these network flow theorems to linear programming.

The presentation adopts the organization of the introduction as follows. In Section 2, we first present a proof of the *Flow Decomposition Theorem* on networks based on linear programming arguments rather than the classical constructive ones. This provides an inspiration for the general case of linear programming. Section 3 establishes our main result based on a specific application of the Dantzig–Wolfe decomposition principle. This is followed in Section 4 by the proof of an *Augmenting Weighted Cycle Theorem* used to derive in Section 5 two alternative optimality conditions for linear programs that are based on the properties of a residual linear problem. We open a discussion in Section 6 which addresses the adaptation to linear programs of the *minimum mean cycle-canceling* algorithm and the design of a column generation based algorithm.

Notation. Vectors and matrices are written in bold face characters. We denote by $\mathbf{0}$ or $\mathbf{1}$ a vector with all zero or one entries of appropriate contextual dimensions.

2. A decomposition theorem for network flow problems

Consider the capacitated minimum cost flow problem (CMCF) on a directed graph $G = (N, A)$, where N is the set of nodes associated with an assumed balanced set $b_i,\ i \in N$, of supply or demand defined respectively by a positive or negative value such that $\sum_{i\in N} b_i = 0$, A is the set of arcs of cost $\mathbf{c} := [c_{ij}]_{(i,j)\in A}$, and

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$\mathbf{x} := [x_{ij}]_{(i,j)\in A}$ is the vector of lower and upper bounded flow variables. An arc-flow formulation of CMCF, where dual variables $\pi_i,\ i \in N$, appear in brackets, is given by

$$\begin{aligned} z^\star_{\text{CMCF}} := \min \quad & \sum_{(i,j)\in A} c_{ij}x_{ij} \\ \text{s.t.} \quad & \sum_{j:(i,j)\in A} x_{ij} - \sum_{j:(j,i)\in A} x_{ji} = b_i, \quad [\pi_i] \quad \forall i \in N \\ & \ell_{ij} \le x_{ij} \le u_{ij}, \qquad \forall (i,j) \in A. \end{aligned} \tag{1}$$

When right-hand side $\mathbf{b} := [b_i]_{i\in N}$ is the null vector, formulation (1) is called a *circulation problem*. The *Flow Decomposition Theorem* for network solutions is as follows.

Theorem 1 (*[1, Theorem 3.5 and Proposition 3.6]*). *Any feasible solution $\mathbf{x}^0$ to CMCF (1) can be represented as a combination of directed path and cycle flows – though not necessarily uniquely – with the following properties:*

(a) *Every path with positive flow connects a supply node to a demand node.*
(b) *At most $|A|+|N|$ paths and cycles have positive flow among which at most $|A|$ cycles.*
(c) *A circulation $\mathbf{x}^0$ is restricted to at most $|A|$ cycles.*

Proof. The proof of the above theorem traditionally relies on a constructive argument. We propose an alternative one based on the application of the Dantzig–Wolfe decomposition principle [2]. The network problem is first converted into a circulation problem, partitioning the set of nodes N in three subsets: supply nodes in $S := \{i \in N \mid b_i > 0\}$, demand nodes in $D := \{i \in N \mid b_i < 0\}$, and transshipment nodes in $N \setminus \{S \cup D\}$ for which $b_i = 0,\ i \in N$. Supplementary nodes s and t are added to N for a convenient representation of the circulation problem together with zero-cost arc sets $\{(s,i) \mid i \in S\}$, $\{(i,t) \mid i \in D\}$, and arc (t,s). Supply and demand requirements are transferred on the corresponding arcs, that is, $\ell_{si} = u_{si} = b_i,\ i \in S$, and $\ell_{it} = u_{it} = -b_i,\ i \in D$. Let $G^+ = (N^+, A^+)$ be the new network on which is defined the circulation problem.

Flow conservation equations for nodes in N^+ together with the nonnegativity requirements on arcs in A^+ portray a circulation problem with no upper bounds. These define the domain $\mathcal{SP}$ of the Dantzig–Wolfe subproblem whereas lower and upper bound constraints remain in the master problem. By the Minkowski–Weyl's theorem (see [9,3]), there is a vertex-representation for the domain $\mathcal{SP}$. The latter actually forms a cone that can be described in terms of a single extreme point (the null flow vector) and a finite number of extreme rays, see Lübbecke and Desrosiers [7] for additional representation applications.

These extreme rays are translated to the original network upon which is done the unit flow interpretation in terms of paths and cycles. For an extreme ray with $x_{ts} = 1$, we face an external cycle in G^+, that is, a *path* within G from a supply node to a demand node, while an extreme ray with $x_{ts} = 0$ implies an internal cycle in G^+, that is, a *cycle* within G. Furthermore, the extreme ray solutions to $\mathcal{SP}$ naturally satisfy the flow conservation constraints and therefore respect the directed nature of G. Paths and cycles are therefore understood to be directed even though we omit the precision in the spirit of concision.

Let $\mathcal{P}$ and $\mathcal{C}$ be respectively the sets of paths and cycles in G. The null extreme point at no cost can be removed from the Dantzig–Wolfe reformulation as it has no contribution in the constraint set of the master problem. Any nonnull solution $[\mathbf{x}, \mathbf{x}_S, \mathbf{x}_D, x_{ts}]^\intercal$ to $\mathcal{SP}$ can therefore be written as a nonnegative combination of the extreme rays only, that is, in terms of the supply–demand paths $[\mathbf{x}_p, \mathbf{x}_{Sp}, \mathbf{x}_{Dp}, 1]^\intercal,\ p \in \mathcal{P}$, and internal cycles $[\mathbf{x}_c, \mathbf{0}, \mathbf{0}, 0]^\intercal,\ c \in \mathcal{C}$:

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{x}_S \\ \mathbf{x}_D \\ x_{ts} \end{bmatrix} = \sum_{p\in\mathcal{P}} \begin{bmatrix} \mathbf{x}_p \\ \mathbf{x}_{Sp} \\ \mathbf{x}_{Dp} \\ 1 \end{bmatrix} \theta_p + \sum_{c\in\mathcal{C}} \begin{bmatrix} \mathbf{x}_c \\ \mathbf{0} \\ \mathbf{0} \\ 0 \end{bmatrix} \phi_c,$$
$$\theta_p \ge 0,\ \forall p \in \mathcal{P}, \phi_c \ge 0,\ \forall c \in \mathcal{C}. \tag{2}$$

Define $c_p = \mathbf{c}^\intercal \mathbf{x}_p,\ p \in \mathcal{P}$, as the cost of a path and $c_c = \mathbf{c}^\intercal \mathbf{x}_c,\ c \in \mathcal{C}$, as the cost of a cycle. The Dantzig–Wolfe master problem, an alternative formulation of CMCF (1) written in terms of nonnegative path and cycle θ, ϕ-variables, is given as

$$\begin{aligned} z^\star_{\text{CMCF}} := \quad \min \quad & \sum_{p\in\mathcal{P}} c_p\theta_p + \sum_{c\in\mathcal{C}} c_c\phi_c \\ \text{s.t.} \quad \mathbf{l} \le \quad & \sum_{p\in\mathcal{P}} \mathbf{x}_p\theta_p + \sum_{c\in\mathcal{C}} \mathbf{x}_c\phi_c \le \mathbf{u} \\ & \sum_{p\in\mathcal{P}} \mathbf{x}_{Sp}\theta_p = \mathbf{b}_S \\ & \sum_{p\in\mathcal{P}} \mathbf{x}_{Dp}\theta_p = -\mathbf{b}_D \\ & \theta_p \ge 0,\ \forall p \in \mathcal{P},\ \phi_c \ge 0,\ \forall c \in \mathcal{C}. \end{aligned} \tag{3}$$

The rest of the proof relies on the dimension of any basis representing a feasible solution $\mathbf{x}^0$ to (1). The latter can be expressed in terms of the change of variables in (2) and satisfies the system of equality constraints in (3):

$$\begin{aligned} & \sum_{p\in\mathcal{P}} \mathbf{x}_p\theta_p + \sum_{c\in\mathcal{C}} \mathbf{x}_c\phi_c = \mathbf{x}^0 \\ & \sum_{p\in\mathcal{P}} \mathbf{x}_{Sp}\theta_p = \mathbf{b}_S \\ & \sum_{p\in\mathcal{P}} \mathbf{x}_{Dp}\theta_p = -\mathbf{b}_D \\ & \theta_p \ge 0,\ \forall p \in \mathcal{P},\ \phi_c \ge 0,\ \forall c \in \mathcal{C}. \end{aligned} \tag{4}$$

Since any basic solution to (4) involves at most $|A| + |S| + |D|$ nonnegative θ, ϕ-variables, there exists a representation for $\mathbf{x}^0$ that uses at most $|A| + |N|$ path and cycle variables, among which at most $|A|$ cycles (ϕ-variables). In the case of a circulation problem for which $\mathbf{b} = \mathbf{0}$, there are no paths involved (no θ-variables) and $\mathbf{x}^0$ can be written as a combination of at most $|A|$ cycles. □

3. A decomposition theorem for linear programs

In this section, we generalize Theorem 1 to the feasible solutions of a linear program (*LP*). Although it is usually frowned upon, we warn the reader that we reuse some of the same notations previously seen in networks. While the semantics are a little bit distorted, we wish to retain the ideas attached to them. The proof again relies on a specific Dantzig–Wolfe decomposition. Consider the following *LP* formulation with lower and upper bounded variables:

$$\begin{aligned} z^\star := \quad \min \quad & \mathbf{c}^\intercal\mathbf{x} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} = \mathbf{b}, \qquad [\boldsymbol{\pi}] \\ & \mathbf{l} \le \mathbf{x} \le \mathbf{u}, \end{aligned} \tag{5}$$

where $\mathbf{x}, \mathbf{c}, \mathbf{l}, \mathbf{u} \in \mathbb{R}^n$, $\mathbf{b} \in \mathbb{R}^m$, $\mathbf{A} \in \mathbb{R}^m \times \mathbb{R}^n$, and $m \le n$. Without loss of generality, we also assume that right-hand side vector $\mathbf{b} \ge \mathbf{0}$. If $\mathbf{b} = \mathbf{0}$, we face a homogeneous system of constraints. The vector of dual variables $\boldsymbol{\pi} \in \mathbb{R}^m$ associated with the equality constraints appears within brackets. In order to perform our specific decomposition, we introduce a vector of nonnegative variables $\mathbf{v} \in \mathbb{R}^m$ and rewrite *LP* (5), splitting the constraints in two subsets: