Operations Research Letters 42 (2014) 208–212

Contents lists available at ScienceDirect

Operations Research Letters

journal homepage: www.elsevier.com/locate/orl

Optimal retirement strategy with a negative wealth constraint

Seyoung Park, Bong-Gyu Jang *

Department of Industrial and Management Engineering, POSTECH, 77 Cheongam-Ro, Nam-Gu, Pohang, Gyeongbuk, 790-784, Republic of Korea

ARTICLE INFO

Article history:
Received 8 October 2013
Received in revised form
1 February 2014
Accepted 25 February 2014
Available online 3 March 2014

Keywords:
Optimal stopping
Optimal retirement
Optimal investment
Optimal portfolio
Wealth constraint

ABSTRACT

This paper investigates an optimal consumption, portfolio, and retirement time choice problem of an individual with a negative wealth constraint. We obtain analytical results of the optimal consumption, investment, and retirement behaviors and discuss the effect of the negative wealth constraint on the optimal behaviors. We find that, as an individual can borrow more with better credit, she is more likely to retire at a higher wealth level, to consume more, and to invest more in risky assets.

1. Introduction

We study an optimal consumption and portfolio selection problem of an individual who wants to voluntarily retire someday in the future. This problem has been considered as one of the most important problems in financial decision making, and technically it is a representative example of a mixture of the conventional consumption/portfolio selection problems (see, e.g., Merton [7]) and an optimal stopping problem. Many researchers have dealt with problems of such kind: particularly, Farhi and Panageas [3] solved an optimal consumption and portfolio selection problem of an individual who wants to choose her optimal retirement time, which can be formulated as an optimal stopping problem.

On the other hand, it is well-known that imposing some wealth constraints in the conventional problems makes them much complicated. Recently, some pioneers such Dybvig and Liu [2] and Jang et al. [5] can successfully figure out their analytic solution under the conditions that an individual can choose her retirement time and her behaviors are restricted by a nonnegative wealth constraint. They found that such constraint might have a significant impact on individual's optimal consumption/investment and retirement behaviors.

However, as far as we know, none of the existing literature addresses an optimal retirement and portfolio selection problem in the presence of a *negative* wealth constraint. In this paper we consider a more realistic economic situation where an individual can borrow some money (specifically, up to some proportion of future income) with her good credit. Our model can include the problem with nonnegative wealth constraint as an extreme case.

This paper contributes to the literature by providing analytical results of optimal consumption, investment, and retirement strategies of an individual in the presence of a negative wealth constraint. Technically, we develop a new convex–duality method for solving the optimal retirement problem. Exploiting the method, we get an analytic solution of the variational inequality which is equivalent to our problem. Subsequently, we show that individual's optimal behaviors are significantly affected by the negative wealth constraint. Specifically, she would retire at a higher wealth level, consume more, and invest more in risky assets, if she can borrow more.

2. The basic model

2.1. Financial market, income, and individual's utility preference

We consider a financial market in which an individual can trade two broad classes of assets: a bond (or a risk-free asset) and a stock (or a risky asset). The bond price B_t satisfies the relationship of

$$dB_t = rB_t dt,$$

where a risk-free interest rate r is positive. The stock price S_t follows

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

where $\mu > 0$ is the expected rate of the stock return, $\sigma > 0$ is the stock volatility, and W_t is a standard Brownian motion defined on a suitable probability space.

* Corresponding author. Tel.: +82 54 279 2372; fax: +82 54 279 2870.
E-mail addresses: seyoungdog@postech.ac.kr (S. Park), bonggyujang@postech.ac.kr (B.-G. Jang).

http://dx.doi.org/10.1016/j.orl.2014.02.005
0167-6377/

The individual is assumed to have the following time-additive utility function of Cobb–Douglas type:

$$U\big(l(t), c(t)\big) \equiv \frac{1}{a}\frac{\left(l(t)^{1-a}c_t^a\right)^{1-\gamma^*}}{1-\gamma^*},$$
$$\gamma^* > 0,\ \gamma^* \neq 1,\ \text{and } 0 < a < 1,$$

where $c(t)$ is the consumption rate and $l(t)$ is leisure at time t, and a is a weight for consumption. The individual is currently a full-time wage earner, but she has an option to choose a voluntary retirement time. She enjoys leisure of $l(t) = l$ while she is working and $l(t) = \bar{l}$ $(l < \bar{l})$ after she voluntarily retires. We assume that the wage rate w is constant and, then, the individual gets an income of $I_1 \equiv w(\bar{l} - l) > 0$ per unit time during working status (see Farhi and Panageas [3] for the details). We also assume that she gets a post-retirement income $I_2 > 0$ $(I_1 > I_2)$ per unit time. The assumption of a positive income after retirement reflects the fact that most countries provide the retired with retirement benefits and other public welfare services.

2.2. A retirement problem in the presence of a negative wealth constraint

We define the coefficient of relative risk aversion $\gamma > 0$ as $1 - a(1-\gamma^*)$ and normalize pre-retirement leisure as $l = 1$, and then the utility function during working becomes

$$U_1(c) \equiv U(1, c) = \frac{c^{1-\gamma}}{1-\gamma}.$$

The retirement problem is to maximize the individual's expected utility by controlling consumption, stock investment, and retirement time, or equivalently, to find

$$\Phi(x) \equiv \max_{(c,\pi,\tau)} E\Big[\int_0^\tau e^{-\beta t}U_1\big(c(t)\big)dt + e^{-\beta\tau}\int_\tau^\infty e^{-\beta(t-\tau)}U\big(l(t), c(t)\big)dt\Big],$$

where π is the dollar amount invested in the stock, τ is voluntary retirement (or optimal stopping) time, and $\beta > 0$ is the individual's subjective discount rate.

The wealth process $X(t)$ of the individual follows

$$dX(t) = \begin{cases} \big(rX(t) - c(t) + I_1\big)dt + \pi(t)\sigma(dW(t) + \theta dt), \\ \quad \text{for } 0 \le t < \tau, \\ \big(rX(t) - c(t) + I_2\big)dt + \pi(t)\sigma(dW(t) + \theta dt), \\ \quad \text{for } t \ge \tau, \end{cases}$$

where θ represents the Sharpe ratio $(\mu - r)/\sigma$. We assume that the individual has an initial wealth x. A negative wealth constraint of

$$X(t) \ge k > -\frac{I_1}{r}, \quad \text{for } 0 \le t \le \tau,\ \ k \le 0 \tag{1}$$

is imposed throughout this paper. It allows the individual to borrow money up to k against future income during working. Note that for the case where k equals to 0, it becomes a nonnegative wealth constraint, which is imposed in Farhi and Panageas [3]. We also assume that, after retirement, it is possible for the individual to borrow up to $-I_2/r$, which is equivalent to the present value of her total post-retirement income.

Following Farhi and Panageas [3], we define for $\gamma > 1$

$$K \equiv \left(\bar{l}^{1-a}\right)^{1-\gamma^*}\left(\frac{1}{\eta}\right)^\gamma < 1,$$

where

$$\eta = \frac{\gamma-1}{\gamma}\Big(r + \frac{\theta^2}{2\gamma}\Big) + \frac{\beta}{\gamma} > 0$$

and assume that $\beta - r < \theta^2/2$ to assure that retirement happens with probability one. We can rewrite the individual's objective function (the so-called *value function*) as

$$\Phi(x) = \max_{(c,\pi,\tau)} E\Big[\int_0^\tau e^{-\beta t}U_1\big(c(t)\big)dt + e^{-\beta\tau}U_2\big(X(\tau)\big)\Big], \tag{2}$$

where

$$U_2(z) = K\frac{(z + I_2/r)^{1-\gamma}}{1-\gamma}.$$

In fact, $U_2(z)$ is the value function of the classical consumption/portfolio selection problem (see, e.g., Merton [7]) under the condition that the investor has a constant relative risk aversion type utility preference and gets an income stream I_2 forever.

3. Problem reformulation: dynamic programming approach

We utilize the dynamic programming approach to solve our problem. For a fixed stopping time τ, we define

$$J_\tau(x) \equiv \max_{(c,\pi)} E\Big[\int_0^\tau e^{-\beta t}U_1\big(c(t)\big)dt + e^{-\beta\tau}U_2\big(X(\tau)\big)\Big],$$

then,

$$\Phi(x) = \max_\tau J_\tau(x).$$

We utilize the following variational inequality with respect to (2) as described in Øksendal [8]:

$$\beta\phi(x) - (rx + I_1)\phi'(x) + \frac{\theta^2}{2}\frac{\phi'(x)^2}{\phi''(x)} - \frac{\gamma}{1-\gamma}\{\phi'(x)\}^{1-1/\gamma} \ge 0, \quad \text{and} \quad \phi(x) \ge U_2(x). \tag{3}$$

Note that optimal consumption prior to voluntary retirement $c^*(\tau-)$ is given by

$$c^*(\tau-) = \{\phi'(x)\}^{-1/\gamma},$$

and optimal consumption after retirement $c^*(\tau+)$, which is easily obtained by using the same arguments in Merton [7], is given by

$$c^*(\tau+) = \bar{l}^{(1-a)(1-\gamma^*)/\gamma}\{\phi'(x)\}^{-1/\gamma}.$$

Then we can obtain the following relationship:

$$c^*(\tau+)/c^*(\tau-) = \bar{l}^{(1-a)(1-\gamma^*)/\gamma} = K^{1/\gamma}\eta < 1,$$

which implies that there exists a downward jump in consumption at retirement date (see Farhi and Panageas [3] for the details). From the inequality (3) we establish a free boundary problem with one free boundary $\hat{x}$:

$$\begin{cases} \beta\phi(x) - (rx + I_1)\phi'(x) + \dfrac{\theta^2}{2}\dfrac{\phi'(x)^2}{\phi''(x)} \\ \qquad -\dfrac{\gamma}{1-\gamma}\{\phi'(x)\}^{1-1/\gamma} = 0, \quad k \le x < \hat{x}, \\ \phi(x) = U_2(x), \quad x \ge \hat{x}, \\ \phi(\hat{x}) = U_2(\hat{x}), \\ \phi'(\hat{x}) = U_2'(\hat{x}). \end{cases} \tag{4}$$

The free boundary $\hat{x}$ is called *critical wealth level*, over which it is optimal for an individual to enter voluntary retirement.

If we find an increasing and concave $\phi(x)$, satisfying C^1 and piecewise C^2 conditions, then $\phi(x)$ in (4) is indeed a solution of the variational inequality (3) (see Theorem 4.1). Further, it is straightforward to verify that the solution $\phi(x)$ of (3) is equivalent to the value function $\Phi(x)$ of our problem (see Theorem 10.4.1 in Øksendal [8]).