Operations Research Letters 40 (2012) 153–158

Contents lists available at SciVerse ScienceDirect

Operations Research Letters

journal homepage: www.elsevier.com/locate/orl

Mixed integer linear programming formulations for probabilistic constraints

J.P. Vielma [a,*], S. Ahmed [b], G.L. Nemhauser [b]

[a] Department of Industrial Engineering, University of Pittsburgh, 1048 Benedum Hall, Pittsburgh, PA 15261, United States

[b] H. Milton Stewart School of Industrial and Systems Engineering, Georgia Institute of Technology, 765 Ferst Drive NW, Atlanta, GA 30332-0205, United States

ARTICLE INFO

Article history:
Received 15 February 2011
Accepted 28 November 2011
Available online 27 January 2012

Keywords:
Mixed integer linear programming
Formulations
Probabilistic constraints
Chance constraints

ABSTRACT

We introduce two new formulations for probabilistic constraints based on extended disjunctive formulations. Their strength results from considering multiple rows of the probabilistic constraints together. The properties of the first can be used to construct hierarchies of relaxations for probabilistic constraints, while the second provides computational advantages over other formulations.

1. Introduction

We consider Mixed Integer Linear Programming (MILP) formulations of joint probabilistic or chance constraints for finitely distributed random variables. For arbitrary distributions such constraints have been extensively studied and have many applications (see for example [18,22] and the references within). The discrete distribution case has been studied in [3,7,11,13,15,21] and used in applications in [3,4,12,16,20]. Finite distributions also appear naturally in Sample Average Approximations (SAA) of general probabilistic constraints [14].

We concentrate on the probabilistically constrained set $Q := \{x \in \mathbb{R}^d : \mathbb{P}\big(x \geq \xi\big) \geq 1 - \delta\}$ where $\delta \in (0, 1)$ and ξ is a d-dimensional random vector with finite support $\{\xi^1, \dots, \xi^S\} \subset \mathbb{R}^d_+$ and with $\mathbb{P}(\xi = \xi^s) = 1/S$ for each $s \in \{1, \dots, S\}$. A standard MILP formulation for Q was introduced in [19] and is given by

$$\sum_{s=1}^{S} z_s \leq k, \tag{1a}$$

$$z_s \in \{0, 1\} \quad \forall s \in \{1, \dots, S\} \tag{1b}$$

$$x \geq (1 - z_s)\xi^s \quad \forall s \in \{1, \dots, S\} \tag{1c}$$

where $k := \lfloor \delta S \rfloor$. This formulation uses binary variables $z \in \{0, 1\}^S$ such that $z_s = 1$ if $x \ngeq \xi^s$ and restricts the number of violated $x \geq \xi^s$ inequalities through the cardinality constraint (1a). The Linear Programming (LP) relaxation of formulation (1) can be very weak, so valid inequalities for it have been developed in [11,15]. In addition, a strengthened version of (1) was introduced in [15]. Because of the use of big-M type constraints based on lower bounds on x we denote the original and strengthened versions of (1) as *Big-M* and *Strong Big-M* formulations respectively.

Alternative MILP formulations for Q can be constructed using standard disjunctive programming arguments. Unfortunately, the sizes of the resulting formulations are exponential in S for fixed δ. Although this size can be significantly reduced by using so-called $(1 - \delta)$-efficient points [3,7,18,21], the resulting sizes usually remain exponential (e.g. see Section 5.3 of [24] for an example in which the reduced formulation size is exponential in d). Hence, it is not practical to use these disjunctive formulations directly and they are mostly used as a base for specialized algorithms [3,6–8] or to construct valid inequalities [19,21,23].

Our main contribution is to introduce two new MILP formulations for Q. The first formulation can be used to construct a hierarchy of relaxations for Q and the second one provides a computational advantage over other formulations. In addition, both formulations can consider more than one row of (1c) at a time. To the best of our knowledge, no other existing formulation can do this without assuming a special structure for $\{\xi^1, \dots, \xi^S\}$.

The rest of this paper is organized as follows. In Section 2 we introduce the new formulations and their theoretical properties and in Section 3 we present results of computational experiments that illustrate the strength and effectiveness of existing and new formulations.

2. New formulations

Our new formulations rely on the following disjunctive characterization of the feasible region of (1) when considering only a subset D of the rows of (1c).

* Corresponding author.
E-mail addresses: jvielma@pitt.edu (J.P. Vielma), sahmed@isye.gatech.edu (S. Ahmed), gnemhaus@isye.gatech.edu (G.L. Nemhauser).

0167-6377/$ – see front matter © 2012 Elsevier B.V. All rights reserved.
doi:10.1016/j.orl.2012.01.007

For $D \subset \{1, \dots, d\}$ let $x_D = (x_i)_{i\in D}$, $\xi^s_D = (\xi^s_i)_{i\in D}$ and define $Q^D := \left\{(x, z) \in \mathbb{R}^D \times \{0, 1\}^S : \sum_{s=1}^S z_s \le k,\ x_D \ge (1 - z_s)\xi^s_D\ \forall s \in \{1, \dots, S\}\right\}$. Additionally for $D \subset \{1, \dots, d\}$ and $g \in \mathbb{R}^D$ let $v_D(g) = \{s \in \{1, \dots, S\} : g \not\ge \xi^s_D\}$ be the set of scenarios for which g violates constraint $g \ge \xi^s_D$ and define $Q^D_g := \left\{(x, z) \in \mathbb{R}^D \times \{0, 1\}^S : x_D \ge g, z_s = 1\ \forall s \in v_D(g), \sum_{s\notin v_D(g)} z_s \le (k - |v_D(g)|)\right\}$. We then have the following two lemmas whose proofs are straightforward.

Proposition 1. *Let $\mathcal{D} \subset 2^{\{1,\dots,d\}}$ be such that $\bigcup_{D\in\mathcal{D}} D = \{1, \dots, d\}$, then (x, z) satisfies (1) if and only if $(x_D, z) \in Q^D$ for all $D \in \mathcal{D}$.*

Proposition 2. *Let $G_D := \left\{g \in \prod_{j\in D}\{\xi^s_j\}_{s=1}^S : |v_D(g)| \le k\right\}$ where $\prod$ denotes the Cartesian product. Then $Q^D = \bigcup_{g\in G_D} Q^D_g$.*

Using these two propositions we can use standard MILP formulations for disjunctive constraints to obtain reformulations of (1). However, before we give them, we refine the characterization of Q^D in Proposition 2 by reducing the number of points in G_D as follows.

Proposition 3. *Let $\tilde{G}_D := \bigcup_{l=0}^k \left\{g \in \prod_{j\in D}\{\xi^s_j\}_{s=1}^S : |v_D(g)| \le l \text{ and } |v_D(g - q)| > l\ \forall q \in \mathbb{R}^D_+ \setminus \{\mathbf{0}\}.\right\}$ then*

$$Q^D = \bigcup_{g\in\tilde{G}_D} Q^D_g. \tag{2}$$

Proof. For the first inclusion let $(x^0, z^0) \in Q^D$ and for each $j \in D$ let $s_j := \arg\max_{s=1}^S\{\xi^s_j : x^0_j \ge \xi^s_j\}$. Let $g \in \mathbb{R}^D$ be such that $g_j := \xi^{s_j}_j$ for each j. Then $g \in \tilde{G}_D$ and $(x^0, z^0) \in Q^D_g$. The reverse inclusion is direct. □

When D is a singleton $G_D = \tilde{G}_D$, but when $|D| > 1$ $\tilde{G}_D$ can be significantly smaller that G_D. However, $\tilde{G}_D$ is usually a strict superset of the $(1 - \delta)$-efficient points associated with Q^D. These facts are illustrated in the following example.

Example 1. Let $d = 2, S = 4, \xi^1 = (0, 20), \xi^2 = (10, 10), \xi^3 = (20, 0), \xi^4 = (30, 30)$ and $k = 2$. For these data the projection onto the x space of (1) is given by the shaded region in Fig. 1. This depicts the choice of $D = \{1, 2\}$ for which $\xi^i = \xi^i_{\{1,2\}}$ for $i \in \{1, \dots, 4\}$. The figure also shows the projections onto the x_1 and x_2 variables which correspond to the choices of $D = \{1\}$ and $D = \{2\}$ respectively. In these two last cases we can see that $G_D = \tilde{G}_D$ (points surrounded by triangles for $D = \{1\}$ and by squares for $D = \{2\}$). For instance, for $D = \{2\}$ we have that $\left|v_{\{2\}}\left(\xi^2_{\{2\}}\right)\right| = 2$ and $\left|v_{\{2\}}\left(\xi^2_{\{2\}} + \lambda w\right)\right| \ge 3$ for any $\lambda > 0$. In contrast, for $D = \{1, 2\}$ we have $\tilde{G}_D = \{(10, 20), (20, 10), (20, 20), (30, 30)\}$ (points surrounded by circles) while $G_D = \{(10, 20), (20, 10), (20, 20), (30, 30), (10, 30), (20, 30), (30, 10), (30, 20)\}$ (points surrounded by circles and diamonds). In particular, $(20, 20)$ is in $\tilde{G}_D$ because $\left|v_{\{1,2\}}((20, 20))\right| = 1$ and $\left|v_{\{1,2\}}((20, 20) - q)\right| \ge 2$ for any $q \in \mathbb{R}^2_+$ (such as for $q = -u$ and $q = -v$), while $(30, 10)$ is not in $\tilde{G}_D$ because $\left|v_{\{1,2\}}((30, 10))\right| = 2$, but $\left|v_{\{1,2\}}((30, 10) + \lambda h)\right| = 2$ for any sufficiently small $\lambda > 0$. Finally, note that $\tilde{G}_D$ strictly contains the set of $(1 - \delta)$-efficient points which is given by $\{(10, 20), (20, 10), (20, 20)\}$ (points surrounded by hexagons).

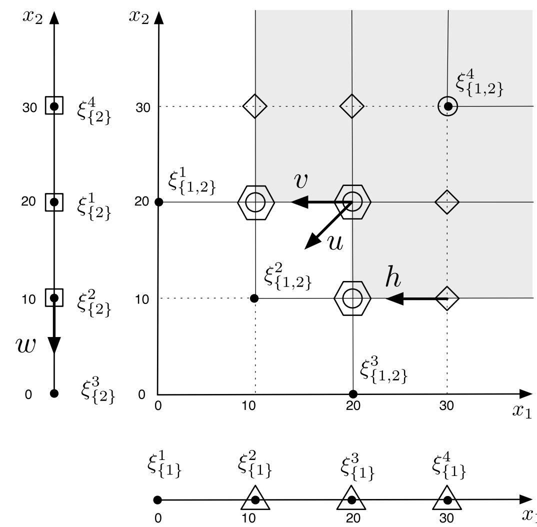

Fig. 1. Feasible region of Example 1.

Using Propositions 1 and 3 we can construct the following two families of formulations for Q.

Proposition 4. *Let $\mathcal{D} \subset 2^{\{1,\dots,d\}}$ be such that $\bigcup_{D\in\mathcal{D}} D = \{1, \dots, d\}$. Then*

$$x_j \ge \sum_{g\in\tilde{G}_D} y^D_g g_j \quad \forall j \in D,\ D \in \mathcal{D} \tag{3a}$$

$$\sum_{g\in\tilde{G}_D} y^D_g = 1 \quad \forall D \in \mathcal{D} \tag{3b}$$

$$y^D_g \in \{0, 1\} \quad \forall g \in \tilde{G}_D,\ D \in \mathcal{D} \tag{3c}$$

$$0 \le z^{D,g}_s \le y^D_g \quad \forall g \in \tilde{G}_D,\ s \in \{1, \dots, S\},\ D \in \mathcal{D} \tag{3d}$$

$$z^{D,g}_s \ge y^D_g \quad \forall g \in \tilde{G}_D,\ s \in v_D(g),\ D \in \mathcal{D} \tag{3e}$$

$$\sum_{s\notin v_D(g)} z^{D,g}_s \le y^D_g(k - |v_D(g)|) \quad \forall g \in \tilde{G}_D,\ D \in \mathcal{D} \tag{3f}$$

$$z_s = \sum_{g\in\tilde{G}_D} z^{D,g}_s \quad \forall s \in \{1, \dots, S\},\ D \in \mathcal{D} \tag{3g}$$

$$z_s \in \{0, 1\} \quad \forall s \in \{1, \dots, S\} \tag{3h}$$

is a valid formulation of Q. A smaller valid formulation is given by

$$x_j \ge \sum_{g\in\tilde{G}_D} y^D_g g_j \quad \forall j \in D,\ D \in \mathcal{D} \tag{4a}$$

$$\sum_{g\in\tilde{G}_D} y^D_g = 1 \quad \forall D \in \mathcal{D} \tag{4b}$$

$$y^D_g \in \{0, 1\} \quad \forall g \in \tilde{G}_D,\ D \in \mathcal{D} \tag{4c}$$

$$\sum_{s=1}^S z_s \le k \tag{4d}$$

$$z_s \in \{0, 1\} \quad \forall s \in \{1, \dots, S\} \tag{4e}$$

$$z_s \ge \sum_{g:s\in v_D(g)} y^D_g \quad \forall s \in \{1, \dots, S\},\ D \in \mathcal{D}. \tag{4f}$$

Proof. For $\mathcal{D} = \{D\}$ we have that (3) with (3a) replaced by

$$x^{D,g}_j \ge y^D_g g_j, \quad x_j = \sum_{g\in\tilde{G}_D} x^{D,g}_j\ \forall g \in \tilde{G}_D,\ j \in D,\ D \in \mathcal{D} \tag{5}$$