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On the sufficiency of finite support duals in semi-infinite linear programming

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ABSTRACT

We consider semi-infinite linear programs with countably many constraints indexed by the natural numbers. When the constraint space is the vector space of all real valued sequences, we show that the finite support (Haar) dual is equivalent to the algebraic Lagrangian dual of the linear program. This settles a question left open by Anderson and Nash (1987). This result implies that if there is a duality gap between the primal linear program and its finite support dual, then this duality gap cannot be closed by considering the larger space of dual variables that define the algebraic Lagrangian dual. However, if the constraint space corresponds to certain subspaces of all real-valued sequences, there may be a strictly positive duality gap with the finite support dual, but a zero duality gap with the algebraic Lagrangian dual.

1. Introduction

We begin with a brief review of notation and basic definitions for semi-infinite linear programs. Let Y be a vector space. The *algebraic dual* of Y is the set of linear functionals with domain Y and is denoted by Y'. Let $\psi \in Y'$. The evaluation of ψ at y is denoted by $\langle y, \psi \rangle$; that is, $\langle y, \psi \rangle = \psi(y)$. We emphasize that the theory presented here deals with algebraic dual spaces and *not* topological dual spaces. Discussion of how our work relates to topological duals appears in Remarks 2.2 and 2.3.

Let P be a convex cone in Y. A convex cone P is *pointed* if and only if $P \cap -P = \{0\}$. In the rest of the paper all convex cones are assumed to be pointed. A pointed convex cone P in Y defines a vector space ordering $\succeq_P$ of Y, with $y \succeq_P y'$ if $y - y' \in P$. The *algebraic dual cone* of P is

$$P' = \left\{ \psi \in Y' : \langle y, \psi \rangle \geq 0 \text{ for all } y \in P \right\}.$$

Elements of P' are called *positive linear functionals* on Y (see for instance, page 17 of [9]). Let $A : X \to Y$ be a linear mapping from vector space X to vector space Y. The *algebraic adjoint* $A' : Y' \to X'$ is a linear operator defined by $A'(\psi) = \psi \circ A$ and satisfies $\langle x, A'(\psi) \rangle = \langle A(x), \psi \rangle$ where $\psi \in Y'$ and $x \in X$. Using this notation, define the primal conic optimization problem

$$\begin{aligned} \inf_{x \in X} \ & \langle x, \phi \rangle \\ \text{s.t. } & A(x) \succeq_P b \end{aligned} \tag{ConLP}$$

where $b \in Y$ and ϕ is a linear functional on X.

Now define the standard algebraic Lagrangian dual for (ConLP).

$$\begin{aligned} \sup_{\psi \in P'} \inf_{x \in X} & \{\langle x, \phi \rangle + \langle b - A(x), \psi \rangle\} \\ &= \sup_{\psi \in P'} \inf_{x \in X} \left\{ \langle x, \phi \rangle + \langle b, \psi \rangle - \langle A(x), \psi \rangle \right\} \\ &= \sup_{\psi \in P'} \left\{ \langle b, \psi \rangle + \inf_{x \in X} \{\langle x, \phi \rangle - \langle A(x), \psi \rangle\} \right\} \\ &= \sup_{\psi \in P'} \left\{ \langle b, \psi \rangle + \inf_{x \in X} \{\langle x, \phi \rangle - \langle x, A'(\psi) \rangle\} \right\} \\ &= \sup_{\psi \in P'} \left\{ \langle b, \psi \rangle + \inf_{x \in X} \langle x, \phi - A'(\psi) \rangle \right\}. \end{aligned}$$

Since $x \in X$ is unrestricted, if $\phi - A'(\psi)$ is not the zero linear functional on X, then the inner minimization goes to negative infinity, so require $\phi - A'(\psi) = \theta_X$, where θ_X is the zero linear functional on X. Then the Lagrangian dual of (ConLP) is

$$\begin{aligned} \sup \ & \langle b, \psi \rangle \\ \text{s.t. } & A'(\psi) = \phi \\ & \psi \in P'. \end{aligned} \tag{ConDLP}$$

This problem is called the *algebraic* Lagrangian dual of (ConLP) since the linear functionals ψ that define the dual problem are in Y', which is the algebraic dual of Y.

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Semi-infinite linear programs. Consider the case where $X = \mathbb{R}^n$ and $Y = \mathbb{R}^I$, i.e., the vector space of real-valued functions with domain I where I is an arbitrary (potentially infinite) set. Let $a^1, a^2, \ldots, a^n$ and b be functions in $Y = \mathbb{R}^I$. Let $A : \mathbb{R}^n \to \mathbb{R}^I$ be the linear mapping $x \mapsto (a^1(i)x_1 + a^2(i)x_2 + \cdots + a^n(i)x_n : i \in I)$. Let $\mathbb{R}^I_+$ denote the pointed cone of $u \in \mathbb{R}^I$ such that $u(i) \geq 0$ for all $i \in I$ and let $P = \mathbb{R}^I_+$. With this specification for the vector spaces X and Y, the map A, right hand side b and cone P, problem (ConLP) reduces to the standard semi-infinite linear program

$$\begin{aligned} \inf_{x\in\mathbb{R}^n}\ & \phi^\top x \\ \text{s.t.}\ & \sum_{k=1}^n a^k(i)x_k \geq b(i) \quad \text{for all } i \in I. \end{aligned} \tag{SILP}$$

There is a slight abuse of notation here. When $X = \mathbb{R}^n$, the algebraic dual X' is isomorphic to $\mathbb{R}^n$ so each linear functional $\phi \in X'$ can be mapped to a vector in $\mathbb{R}^n$. Thus, the primal objective function $\langle x, \phi\rangle$ in (ConLP), is replaced by the inner product $\phi^\top x$ with ϕ now treated as a vector in $\mathbb{R}^n$.

Next consider two alternative duals of (SILP): the algebraic Lagrangian dual and the finite support dual due to Haar [8]. Recall that $(\mathbb{R}^I_+)'$ denotes the algebraic dual cone of $P = \mathbb{R}^I_+$. The algebraic Lagrangian dual of (SILP) using (ConDLP) is

$$\begin{aligned} \sup\ & \langle b, \psi\rangle \\ \text{s.t.}\ & A'(\psi) = \phi \\ & \psi \in (\mathbb{R}^I_+)'. \end{aligned} \tag{DSILP}$$

A second dual is derived as follows. Instead of considering every linear functional $\psi \in (\mathbb{R}^I_+)'$ as above, consider a subset of these linear functionals, called the *finite support elements*. For $u \in \mathbb{R}^I$, the *support* of u is the set $\text{supp}(u) = \{i : u(i) \neq 0\}$. The subspace $\mathbb{R}^{(I)}$ denotes those functions in $\mathbb{R}^I$ with finite support. Let $\mathbb{R}^{(I)}_+$ denote the pointed cone of $v \in \mathbb{R}^{(I)}$ such that $v(i) \geq 0$ for all $i \in I$. Under the natural embedding of $\mathbb{R}^{(I)}$ into $(\mathbb{R}^I)'$ for $u \in \mathbb{R}^I$ and $v \in \mathbb{R}^{(I)}$, write $\langle u, v\rangle = \sum_{i\in I} u(i)v(i)$. The latter sum is well-defined since v has finite support. Under this embedding, $\mathbb{R}^{(I)}_+$ is a subset of $(\mathbb{R}^I_+)'$. Moreover, under this embedding, $A' : (\mathbb{R}^I)' \to X' (= \mathbb{R}^n)$ restricted to $\mathbb{R}^{(I)}$ becomes the map $A'(v) = (\sum_{i\in I} a^k(i)v(i))^n_{k=1}$. The *finite support dual* is

$$\begin{aligned} \sup\ & \sum_{i\in I} b(i)v(i) \\ \text{s.t.}\ & \sum_{i\in I} a^k(i)v(i) = \phi_k, \quad k = 1, \ldots, n \\ & v \in \mathbb{R}^{(I)}_+. \end{aligned} \tag{FDSILP}$$

The finite support dual (FDSILP) is restricted to the linear functionals ψ that can be mapped to $v \in \mathbb{R}^{(I)}_+$ under the standard embedding of $\mathbb{R}^{(I)}$ into $(\mathbb{R}^I)'$. Therefore $v(\text{FDSILP}) \leq v(\text{DSILP})$ where the optimal value of optimization problem $(*)$ is denoted by $v(*)$. This leads naturally to the following question.

Question 1. Is it possible that $v(\text{SILP}) = v(\text{DSILP})$ and yet $v(\text{SILP}) > v(\text{FDSILP})$? In other words, can there exist a duality gap between the primal and its finite support dual that is closed by considering the algebraic Lagrangian dual?

This question is significant for the study of semi-infinite linear programming for at least two reasons. First, most duality theory has been developed for the finite support dual [3,5,6,10,12,14]. Moreover, the only other dual given significant attention in the literature is the "continuous dual" (see for instance [4,7]) and this dual shares many of the same duality properties as the finite support dual. Indeed, as stated by Goberna in [4]: "all known duality theorems guaranteeing the existence of a zero duality gap have the same hypotheses for both dual problems [the finite support dual and the continuous dual]". He even goes so far to say that the finite support dual and the continuous dual are "equivalent in practice".

Second, the algebraic Lagrangian dual is notoriously challenging to characterize and work with. Indeed, to the author's knowledge, little has been said about the algebraic dual in the semi-infinite programming literature (only a few studies mention it, and they do not draw conclusions about its connection with the finite support dual [2,13]).

To the authors' knowledge Question 1 has not been settled for $I = \mathbb{N}$, i.e., semi-infinite linear programs with countably many constraints. Indeed, on page 66 of Anderson and Nash's seminal work [2] they write: "It seems to be hard, if not impossible, to find examples of countable semi-infinite programs which have a duality gap in this formulation [the finite support dual], but have no duality gap when we take W to be a wider class of sequences" where W refers to the vector space of dual variables. In our notation, $W = (\mathbb{R}^\mathbb{N})'$ in (DSILP) and $W = \mathbb{R}^{(\mathbb{N})}$ in (FDSILP). Semi-infinite linear programs with countably many constraints have been well-studied in the literature, particularly from the perspective of duality [3,10,11]. In fact, one can even show that, theoretically, there is no loss in generality in considering the countable case. Theorem 2.3 in [11] shows that every semi-infinite linear program with uncountably many constraints can be equivalently reposed over a countable subset of the original constraints.

The main result of this paper (Theorem 2.4) proves that the answer to Question 1 is no for the case of $I = \mathbb{N}$, settling Anderson and Nash's open question. We show that $v(\text{DSILP}) = v(\text{FDSILP})$ by establishing that (DSILP) and (FDSILP) are equivalent programs.

However, there is a subtlety in Question 1 to keep in mind for semi-infinite linear programs with countably many constraints. In the above discussion, a semi-infinite linear program with countably many constraints was cast as an instance of (ConLP) with $X = \mathbb{R}^n$, $Y = \mathbb{R}^\mathbb{N}$, $A : X \to Y$ defined by $A(x) = (a^1(i)x_1 + a^2(i)x_2 + \cdots + a^n(i)x_n : i \in I)$, and $P = \mathbb{R}^\mathbb{N}_+$. Then (DSILP) was formed using (ConDLP). However, if the functions $a^1, a^2, \ldots, a^n$ and b lie in a subspace $\mathcal{V} \subseteq \mathbb{R}^\mathbb{N}$, then we may use $Y = \mathcal{V}$ and $P = \mathcal{V} \cap \mathbb{R}^\mathbb{N}_+$ to write the semi-infinite linear program as an instance of (ConLP). The corresponding (ConDLP) is

$$\begin{aligned} \sup\ & \langle b, \psi\rangle \\ \text{s.t.}\ & A'(\psi) = \phi \\ & \psi \in (\mathcal{V} \cap \mathbb{R}^\mathbb{N}_+)' \end{aligned} \tag{DSILP($\mathcal{V}$)}$$

where $(\mathcal{V} \cap \mathbb{R}^\mathbb{N}_+)' \subseteq \mathcal{V}'$ is the dual cone of $P = \mathcal{V} \cap \mathbb{R}^\mathbb{N}_+$, which lies in the algebraic dual of $\mathcal{V}$.

It is quite possible that a positive linear functional defined on $(\mathcal{V} \cap \mathbb{R}^\mathbb{N}_+)'$ cannot be extended to $(\mathbb{R}^N_+)'$. This implies (DSILP) (with $I = \mathbb{N}$) may have a smaller value than (DSILP($\mathcal{V}$)), i.e., $v(\text{DSILP}) < v(\text{DSILP}(\mathcal{V}))$. In this context, the following question is a natural extension of Question 1.

Question 2. Is it possible that $v(\text{SILP}) = v(\text{DSILP}(\mathcal{V}))$ and $v(\text{SILP}) > v(\text{FDSILP}) = v(\text{DSILP})$ when $a^1, \ldots, a^n, b \in \mathcal{V}$ for some subspace $\mathcal{V} \subseteq \mathbb{R}^\mathbb{N}$? In other words, when the constraint space $\mathcal{V}$ lies in a subspace of $\mathbb{R}^\mathbb{N}$, can there exist a duality gap between the primal and its finite support dual (FDSILP), that is closed by considering the algebraic Lagrangian dual defined according to that subspace?

We show in Section 3 that this can happen. More concretely, in Example 3.5 in Section 3, there is a duality gap between (SILP) and the finite support dual (FDSILP). However, if $a^1, \ldots, a^n, b$ are considered as elements of the space of convergent real sequences