



New structural properties of (s, S) policies for inventory models with lost sales

Yanyi Xu^a, Arnab Bisi^b, Maqbool Dada^{c,*}

^a School of Management, Shanghai University, Shanghai, 200444, China

^b Krannert School of Management, Purdue University, West Lafayette, IN 47907, USA

^c Johns Hopkins Carey Business School, Baltimore, MD 21201, USA

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ABSTRACT

We revisit the classical inventory model for which (s, S) policies are optimal. We consider the finite and infinite horizon versions of the lost sales problem. New structural properties are developed for the optimal policy and cost function. These properties are then used to develop computational schemes for the lost sales problem with Erlang demands.

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1. Introduction

We consider finite and infinite horizon cases of the lost sales variant of the classical stochastic inventory problem: At the start of each period the inventory is reviewed to determine if it should be replenished. If an order is placed, the cost consists of a fixed ordering cost K , and a unit variable cost c . In each period demand that cannot be filled is lost at a unit cost l ; if demand is less than the available inventory, the leftover stock is carried to the next period at a unit holding cost h . The objective is to find the inventory policy that minimizes the expected discounted cost. It is well known that this cost is minimized by following an optimal (s, S) policy.

Under an (s, S) policy, the inventory position of a product is reviewed regularly, and, if it is found to be below a threshold s an order is placed to bring the inventory position up to the level S . The choice of s and S is made to minimize the expected cost while taking into account the fixed ordering cost, inventory holding cost, and the cost associated with stockouts. The fundamental work on (s, S) policies is due to Arrow et al. [2] and Karlin [10]. However, the optimality of such a policy, under general conditions, for the finite horizon case with backorders was first established by Scarf [12] using the novel properties of K -convexity. The application of Scarf's seminal result to the infinite horizon case with stationary demand and cost parameters is due to Iglehart [9]. Subsequently, Veinott [14] and Porteus [11] provided alternative proofs under different conditions. Recently, Beyer and Sethi [6] added technical

details missing in [9]. Analogous results on the structure of the optimal policy for the lost sales case are due to Shreve [13]; many details are also provided by Bertsekas [4] and Bensoussan et al. [3].

When it is appropriate to model demand as being discrete, effective computation schemes are available for the infinite horizon case with backorders. Of special interest are the efficient algorithms due to Federgruen and Zipkin [7] and Zheng and Federgruen [16]; the latter essentially reduces the problem to that of evaluating a single policy. Additional insights into such algorithms are offered by Feng and Xiao [8]. While these papers develop algorithms to find the optimal policy that minimizes average cost per unit time for the backorder case when demand is discrete, we compute the optimal policy for the complementary case that minimizes the expected discounted cost for the lost sales case when demand is continuous.

We begin our development in Section 2 by formulating the finite horizon version of the model and developing some new structural properties that are presented in Theorem 1 for the finite horizon case and in Theorem 2 for the infinite horizon case. While these properties hold without any distributional assumption on demand, additional analysis, as presented in Theorem 3 of Section 3, reveals that when demand is modeled as having an Erlang distribution, the cost function has appealing special structures. These structures lead to a computational scheme for finding the optimal policy for the infinite horizon problem with lost sales.

2. New structural properties

In the finite horizon model there are T periods with the last period labeled as 1 and the first period labeled as T . The demand in each period is described by a continuous random variable that

* Corresponding author.

E-mail addresses: yxu@shu.edu.cn (Y. Xu), abisi@purdue.edu (A. Bisi), mdada1@jhu.edu (M. Dada).

is independently and identically distributed. In developing the model, we will use the following notation:

Cost and model parameters

K = fixed setup cost, c = unit variable cost, h = unit inventory holding cost
 l = unit lost sales cost ($l > c$), b = unit backorder cost
 α = discount factor ($0 < \alpha < 1$), T = time horizon

Demand information

ξ_t = random observation of demand in period t , $t = 1, 2, \dots, T$
 $f(\cdot)$ = probability density function (PDF) of demand in each period (we assume $f(x) > 0$ for $x > 0$)
 $F(\cdot)$ = cumulative distribution function (CDF) of demand in each period

Decision variables

s_t = optimal reorder level in period t
 S_t = optimal order-up-to level in period t

Cost functions

$L(\cdot)$ = one period inventory holding and shortage penalty cost function
 $V_t(x)$ = total minimal expected cost from period t onwards ($t = 1, \dots, 2, 1$), given that the on-hand inventory at the beginning of period t is x
 $G_t(y)$ = total expected cost from period t onwards after inventory level is increased to y
 $G(y|s, S)$ = total expected discounted cost for an infinite horizon problem with stationary reorder point s and order-up-to level S after inventory level is increased to y

Other useful functions

$\delta(z) = \begin{cases} 1 & z > 0 \\ 0 & z = 0 \end{cases}$, the indicator function for ordering decisions
 $x^+ = \max\{x, 0\}$

Since the planning horizon consists of T periods, it is convenient to include a mechanism to “settle accounts” at the end of the planning horizon. Since all excess demand is lost, the state variable x at time 0 must be non-negative. Specifically, we assume that leftover inventory is sold at unit value w ($w \leq c$); if w is positive the inventory is salvaged, and, if w is negative, disposal costs are incurred. Hence, we can conclude that:

$$V_0(x) = -wx \quad (x \geq 0). \quad (1)$$

As in the classical treatment of the (s, S) inventory model, we assume instantaneous delivery of orders, which makes it convenient to define the one period cost, which is time independent because all parameters are stationary. Given that an order is placed and received to bring the inventory level up to y , it is given by:

$$L(y) = h \int_0^y (y - \xi) f(\xi) d\xi + l \int_y^{+\infty} (\xi - y) f(\xi) d\xi. \quad (2)$$

And, given that an optimal policy is followed for the remaining periods, the expected discounted cost with y units of inventory after ordering in period t is given by:

$$G_t(y) = cy + L(y) + \alpha E[V_{t-1}(y - \xi_t)^+], \quad t = 1, 2, \dots, T. \quad (3)$$

Hence, $G_t(y)$ refers to the total expected cost from period t onwards, when the inventory level is increased to y . Thus,

$$V_t(x) = -cx + \min\{K\delta(y - x) + G_t(y) : y \geq x\}, \quad (4)$$

represents the minimal total discounted cost from period t to the last period when the initial on-hand inventory equals x . From Shreve [13], we know that both $G_t(\cdot)$ and $V_t(\cdot)$ are K -convex; hence there exist (s_t, S_t) which minimize $V_t(x)$: if the initial on-hand inventory $x \leq s_t$, the retailer will order $S_t - x$; if $x > s_t$, no order is placed.

In order to find an upper bound for reorder levels, we define

$$G_0(y) = cy + L(y) - \alpha c \int_0^y (y - \xi) f(\xi) d\xi, \quad (5)$$

which can be interpreted as a single-period newsvendor model, where all leftovers are fully refunded with purchase cost c . Since $G_0(y)$ is strictly convex for $y > 0$, there exists a unique number $S_0 (> 0)$ at which $G_0(\cdot)$ attains its local minimum. As shown in

the following theorem, S_0 provides an upper bound for $\tilde{s}_t = \max\{s_t, s_{t-1}, \dots, s_1\}$, the maximum of reorder levels:

Theorem 1. In a T -period problem, for period t , $t = 1, 2, \dots, T$, we have

- (1.1) $s_t \geq s_1, \forall t \geq 2$;
- (1.2) $\tilde{s}_t < S_0$;
- (1.3) $G'_t(y) < 0, \forall y \in (0, \tilde{s}_t]$;
- (1.4) $S_t > \tilde{s}_{t-1}$;
- (1.5) $G'_t(y) \geq c - l, \forall y > 0$.

Proof. Before proving the theorem, we derive a few relationships that will facilitate the proofs. Note that

$$\begin{aligned} G_1(y) &= cy + L(y) + \alpha E[V_0((y - \xi)^+)] \\ &= cy + L(y) - \alpha w \int_0^y (y - \xi) f(\xi) d\xi, \end{aligned} \quad (6)$$

which is the cost function of a single-period newsvendor model, hence is convex; and

$$\begin{aligned} G'_1(y) &= c - l + (h + l - \alpha w)F(y) = 0 \Rightarrow S_1 \\ &= F^{-1}\left(\frac{l - c}{h + l - \alpha w}\right). \end{aligned} \quad (7)$$

Also, for $G_0(y)$ defined in (5), we have

$$\begin{aligned} G'_0(y) &= c - l + (h + l - \alpha c)F(y) = 0 \Rightarrow S_0 \\ &= F^{-1}\left(\frac{l - c}{h + l - \alpha c}\right). \end{aligned} \quad (8)$$

Comparing (7) and (8), we have

$$\begin{aligned} G'_0(y) - G'_1(y) &= \alpha(w - c)F(y) \leq 0, \\ \text{i.e., } G'_0(y) &\leq G'_1(y) \text{ since } w \leq c. \end{aligned} \quad (9)$$

Moreover, since $F^{-1}(\cdot)$ is monotone increasing, we have $S_1 \leq S_0$, and $s_1 < S_1 \leq S_0$.

Second, from the properties of K -convexity, we know

$$\begin{aligned} V_{m+1}(x) &= \begin{cases} -cx + G_{m+1}(S_{m+1}) + K & \text{if } 0 \leq x \leq S_{m+1}, \\ -cx + G_{m+1}(x) & \text{if } x > S_{m+1}; \end{cases} \\ &\text{for } m = 0, 1, 2, \dots, T - 1. \end{aligned}$$

Moreover, if $0 \leq y \leq s_m$,

$$\begin{aligned} G_{m+1}(y) &= cy + L(y) + \alpha E[V_m((y - \xi)^+)] \\ &= cy + L(y) + \alpha \left[K + G_m(S_m) - c \int_0^y (y - \xi) f(\xi) d\xi \right] \\ &= G_0(y) + \alpha[K + G_m(S_m)]. \end{aligned} \quad (10)$$

((10) follows from the fact that: if $y \leq s_m$, then $(y - \xi)^+ \leq s_m$; consequently, an order of $S_m - (y - \xi)^+$ would be placed in period m .)

And, if $y \geq s_m$, then

$$\begin{aligned} G_{m+1}(y) &= cy + L(y) + \alpha E[V_m((y - \xi)^+)] \\ &= cy + L(y) + \alpha \int_0^{y-s_m} [G_m(y - \xi) - c(y - \xi)] f(\xi) d\xi \\ &\quad + \alpha \left[(K + G_m(S_m)) \int_{y-s_m}^y f(\xi) d\xi - c \int_{y-s_m}^y (y - \xi) f(\xi) d\xi \right] \\ &\quad + \alpha \left[(K + G_m(S_m)) \int_y^{+\infty} f(\xi) d\xi \right] \\ &= G_0(y) + \alpha \int_0^{y-s_m} G_m(y - \xi) f(\xi) d\xi \\ &\quad + \alpha (K + G_m(S_m)) (1 - F(y - s_m)). \end{aligned} \quad (11)$$

((11) is derived from three contingencies: (1) if $s_m \leq (y - \xi)$, then

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