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Gap inequalities for non-convex mixed-integer quadratic programs

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ABSTRACT

Laurent and Poljak introduced a very general class of valid linear inequalities, called gap inequalities, for the max-cut problem. We show that an analogous class of inequalities can be defined for general non-convex mixed-integer quadratic programs. These inequalities dominate some inequalities arising from a natural semidefinite relaxation.

1. Introduction

A popular and very powerful approach to solving $\mathcal{NP}$-hard optimisation problems is to formulate them as integer or mixed-integer programs, and then derive strong valid linear inequalities, which can be used within cutting-plane or branch-and-cut algorithms (see, e.g., [7,8]).

Laurent and Poljak [18] introduced an intriguing class of inequalities, called *gap inequalities*, for a combinatorial optimisation problem known as the max-cut problem. They showed that the gap inequalities not only dominate some inequalities arising from the well-known semidefinite programming (SDP) relaxation of the max-cut problem, but also include many other known inequalities as special cases.

In this paper, we show that the idea underlying the gap inequalities can be adapted, in a natural way, to yield gap inequalities for non-convex *Mixed-Integer Quadratic Programs* (MIQPs). Following Laurent and Poljak, we show that these inequalities dominate some inequalities arising from a natural SDP relaxation of non-convex MIQPs. This leads us to conjecture that the generalised gap inequalities are likely to make useful cutting planes for such problems, provided that effective heuristics for generating them can be developed.

The structure of the paper is as follows. In Section 2, we review the relevant literature. In Section 3, we derive gap inequalities for unconstrained 0–1 quadratic programs. Then, in Section 4, we derive them for general non-convex MIQPs.

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2. Literature review

For surveys on the max-cut problem and related problems, we refer the reader to [11,16]. Here, we present only what is needed for the sake of exposition.

A set F of edges in an undirected graph is called an *edge cutset*, or simply *cut*, if there exists a set S of vertices such that an edge is in F if and only if exactly one of its end-vertices is in S. It is known that a vector $y \in \{0,1\}^{\binom{n}{2}}$ is the incidence vector of a cut in the complete graph K_n if and only if it satisfies the following *triangle inequalities*:

$$y_{ij} + y_{ik} + y_{jk} \le 2 \quad (1 \le i < j < k \le n) \tag{1}$$

$$y_{ij} - y_{ik} - y_{jk} \le 0 \quad (1 \le i < j \le n;\ k \ne i, j). \tag{2}$$

The cut polytope, which we will denote by CUT_n, is the convex hull in $\mathbb{R}^{\binom{n}{2}}$ of such incidence vectors (Barahona & Mahjoub [3]). That is,

$$\text{CUT}_n = \text{conv}\left\{ y \in \{0,1\}^{\binom{n}{2}} : (1), (2) \text{ hold} \right\}.$$

This polytope has been studied in great depth; see again [11,16].

The well-known SDP relaxation of the max-cut problem (see [13,17,23]) is based on the following fact (which is easily proved). Let M be the matrix of order n with $M_{ii} = 1$ for $1 \le i \le n$ and $M_{ij} = M_{ji} = 1 - 2y_{ij}$ for all $1 \le i < j \le n$. Then M is positive semidefinite (psd).

As pointed out by Laurent and Poljak [17], M is psd if and only if y satisfies the following infinite family of linear inequalities:

$$\sum_{1 \le i < j \le n} \alpha_i \alpha_j y_{ij} \le \sigma(\alpha)^2/4 \quad (\forall \alpha \in \mathbb{R}^n), \tag{3}$$

where $\sigma(\alpha)$ denotes $\sum_{i \in V} \alpha_i$. We call the inequalities (3) *psd* inequalities.

Observe that, if $\alpha \in \mathbb{Z}^n$ and $\sigma(\alpha)$ is odd, then the psd inequalities can be strengthened by rounding down the right-hand side to the nearest integer. These 'rounded' psd inequalities collectively dominate all of the psd inequalities [11], and have been studied in [2,11,12,20]. They include as special cases the *hypermetric* inequalities of Deza [10] and Kelly [15], the *triangle* inequalities (1)–(2), and the *odd clique* inequalities of Barahona and Mahjoub [3].

The *gap* inequalities, derived by Laurent and Poljak [18], are even stronger and more general than the rounded psd inequalities. They take the form:

$$\sum_{1\le i<j\le n} \alpha_i\alpha_j y_{ij} \le \left(\sigma(\alpha)^2 - \gamma(\alpha)^2\right)/4 \quad (\forall \alpha \in \mathbb{Z}^n), \tag{4}$$

where

$$\gamma(\alpha) := \min\left\{|z^T\alpha| : z \in \{\pm 1\}^n\right\}$$

is the so-called *gap* of α. In [18], it is shown that every gap inequality defines a proper face of CUT_n, though not necessarily a facet. Equivalently, the right-hand side of (4) is best possible. On the other hand, they point out that computing $\gamma(\alpha)$ is $\mathcal{NP}$-hard, since testing whether $\gamma(\alpha) = 0$ is equivalent to the *partition problem*, proven to be $\mathcal{NP}$-complete by Karp [14].

Finally, we mention the *Boolean quadric polytope*, which was introduced by Padberg [22], in the context of unconstrained 0–1 quadratic programming. The Boolean quadric polytope of order n, which we denote by BQP_n, is defined as:

$$\mathrm{BQP}_n = \mathrm{conv}\left\{(x, X) \in \{0,1\}^{n+\binom{n}{2}} : X_{ij} = x_i x_j \ (1 \le i < j \le n)\right\}.$$

(Note that the X_{ij} variables are not defined when $i = j$. There is no need, given that $x_i = x_i^2$ when x_i is binary.) It is known [4,9,22] that CUT_{n+1} can be mapped onto BQP_n using the following linear transformation, known as the *covariance mapping*:

$$x_i = y_{i,n+1} \quad (1 \le i \le n)$$
$$X_{ij} = (y_{i,n+1} + y_{j,n+1} - y_{ij})/2 \quad (1 \le i < j \le n).$$

As a result, if $a^T y \le b$ is any valid inequality for CUT_{n+1}, the inequality

$$\sum_{i=1}^n \left(\sum_{j\in\{1,\dots,n+1\}\setminus\{i\}} a_{ij}\right) x_i - 2\sum_{1\le i<j\le n} a_{ij}X_{ij} \le b$$

is valid for BQP_n. We will use this fact in the next section.

3. From max-cut to unconstrained 0–1 QP

Given any vector $\alpha = (\alpha_1, \dots, \alpha_{n+1})^T \in \mathbb{R}^{n+1}$, one can form a psd inequality for CUT_{n+1}. Now, if the covariance mapping is applied to the psd inequality, one obtains a valid inequality for BQP_n that can be written in the following form:

$$\sum_{i=1}^n \alpha_i(\alpha_i - \sigma(\alpha))x_i + 2\sum_{1\le i<j\le n} \alpha_i\alpha_j X_{ij} + \sigma(\alpha)^2/4 \ge 0. \tag{5}$$

These valid inequalities for BQP_n were also derived by Sherali and Fraticelli [26] in a different way, using the well-known fact [21] that the matrix

$$\begin{pmatrix}1\\x\end{pmatrix}\begin{pmatrix}1\\x\end{pmatrix}^T = \begin{pmatrix}1 & x^T\\ x & xx^T\end{pmatrix}$$

is psd.

Now suppose that $\alpha \in \mathbb{Z}^{n+1}$ and $\sigma(\alpha)$ is odd. Then, one can form a rounded psd inequality for CUT_{n+1}. Applying the covariance mapping again, one finds that the right-hand side of (5) can be increased by 1/4 when α satisfies the stated conditions. We remark that, if we let $\tilde{\alpha}$ denote the truncated vector $(\alpha_1, \dots, \alpha_n)^T$, and β denote $\lfloor\sigma(\alpha)/2\rfloor$, the resulting inequalities for BQP_n can be written in the following form:

$$\sum_{i=1}^n \tilde{\alpha}_i(\tilde{\alpha}_i - 2\beta - 1)x_i + 2\sum_{1\le i<j\le n} \tilde{\alpha}_i\tilde{\alpha}_j X_{ij} + \beta(\beta+1) \ge 0$$
$$(\forall \tilde{\alpha} \in \mathbb{Z}^n, \beta \in \mathbb{Z}). \tag{6}$$

These valid inequalities for BQP_n were also derived by Boros and Hammer [6], again in a different way. Their proof is based on the observation that

$$(\tilde{\alpha}^T x - \beta)(\tilde{\alpha}^T x - \beta - 1) \ge 0$$

whenever x, $\tilde{\alpha}$ and β are integral.

Clearly, the same transformation can be applied to the gap inequalities. The resulting valid inequalities for BQP_n, which are valid for all $\alpha \in \mathbb{Z}^{n+1}$, can be written in the following form:

$$\sum_{i=1}^n \alpha_i(\alpha_i - \sigma(\alpha))x_i + 2\sum_{1\le i<j\le n} \alpha_i\alpha_j X_{ij} + \left(\sigma(\alpha)^2 - \gamma(\alpha)^2\right)/4 \ge 0. \tag{7}$$

We call these inequalities the *gap inequalities for* BQP_n. As far as we know, they have never appeared explicitly before in the literature.

From the above results on the cut polytope and the covariance mapping, it follows that the gap inequalities for BQP_n dominate the Boros–Hammer inequalities (6), which in turn dominate the inequalities (5).

The following proposition shows that gap inequalities of a specific kind dominate all others.

Proposition 1. *For a given vector* $\tilde{\alpha} \in \mathbb{Z}^n$, *let* $S(\tilde{\alpha})$ *be the set of all possible distinct values that* $\tilde{\alpha}^T x$ *can take when x is binary. That is, let*

$$S(\tilde{\alpha}) = \left\{z \in \mathbb{Z} : \exists x \in \{0,1\}^n : \tilde{\alpha}^T x = z\right\}.$$

Let $c = |S(\tilde{\alpha})|$, *and suppose that the elements of* $S(\tilde{\alpha})$ *have been ordered as* $v_1 < v_2 < \cdots < v_c$. *Then, for* $k = 1, \dots, c-1$, *the inequality*

$$\sum_{i=1}^n \tilde{\alpha}_i(\tilde{\alpha}_i - v_k - v_{k+1})x_i + 2\sum_{1\le i<j\le n} \tilde{\alpha}_i\tilde{\alpha}_j X_{ij} + v_k v_{k+1} \ge 0 \tag{8}$$

is a gap inequality for BQP_n. *Moreover, every gap inequality for* BQP_n *is either an inequality of the form* (8), *or dominated by such inequalities.*

Proof. To see that inequality (8) is a special case of inequalities (7), set $\alpha_i = \tilde{\alpha}_i$ for $i = 1, \dots, n$, and set α_{n+1} to $v_k + v_{k+1} - \sigma(\tilde{\alpha})$. This causes $\sigma(\alpha)$ to equal $v_k + v_{k+1}$, and causes $\gamma(\alpha)$ to equal $v_{k+1} - v_k$, which in turn causes (7) to reduce to (8).

To prove dominance, it is helpful to let t denote $\tilde{\alpha}^T x$ and let T denote

$$\sum_{i=1}^n \tilde{\alpha}_i^2 x_i + 2\sum_{1\le i<j\le n} \tilde{\alpha}_i\tilde{\alpha}_j X_{ij}.$$

Then, we make the following three observations:

1. The trivial bounds $0 \le x_i \le 1$ for $i = 1, \dots, n$ imply that $v_1 \le t \le v_c$.
2. The gap inequalities (7) can be written as
$$T \ge \sigma(\alpha)t - \left(\sigma(\alpha)^2 - \gamma(\alpha)^2\right)/4. \tag{9}$$
3. The special gap inequality (8) can be written as
$$T \ge (v_k + v_{k+1})t - v_k v_{k+1}. \tag{10}$$

To complete the proof, we show that the inequalities of the form (10) dominate the inequalities (9). We consider three cases: