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A general framework for cooperation under uncertainty

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ABSTRACT

In this paper, we introduce a general framework for situations with decision making under uncertainty and cooperation possibilities. This framework is based upon a two stage stochastic programming approach. We show that under relatively mild assumptions the associated cooperative games are totally balanced. Finally, we consider several example situations.

1. Introduction

In this paper, we consider situations with multiple players, who choose strategies to influence their expected profits. We assume two decision epochs for an individual player. First, he decides on a strategy to play under uncertainty of the future state of the world, which affects the outcome of the played strategy. After the uncertainty is resolved, the player can take a recourse action that compensates for any adverse effects that might have been experienced as a result of the chosen strategy. The optimal strategies and recourse actions for the players are determined by the solution of a two stage stochastic optimization problem. These players can also cooperate in a coalition. In this case, the players in the coalition coordinate their strategies and recourse actions to maximize their total expected profit.

Many real life situations with decision making under uncertainty can be modeled using two stage stochastic programming. Several applications appeared in the supply chain literature. One example is the analysis of multi-product inventory systems with substitution. A series of papers analyzed these systems with random demand (see [1] and [2]) and random yield (see [3]), where in the first period a production decision is made and after uncertainty is resolved an allocation decision follows. Another application concerns inventory systems with transshipment. Herer and Rashit [4] considered a two-location inventory system with fixed and joint replenishment costs and they developed the properties of optimal decisions. Besides the above applications, Doğru et al. [5] studied a base stock policy for an assemble-to-order system where the products have common components. They developed a heuristic where the stock levels are set by solving a two-stage stochastic program. Moreover, the solution provides a lower-bound for the system performance. Van Mieghem and Rudi [6] introduced a class of models, called newsvendor networks, that provide a framework to study various problems of stochastic capacity investment and inventory management. Their approach is based on a similar two-stage stochastic programming technique as in this paper. All of the papers above, different from ours, assume single ownership of the problem and focus on the determination of optimal decisions or developing effective heuristics. Anupindi et al. [7], Granot and Sošić [8] and Rudi et al. [9] analyzed the performance of decentralized systems, where the centralized (benchmark) performance is given by the solution of a two-stage stochastic program.

In this paper, we provide a general framework for situations in which multiple players collaborate by coordinating their strategies and recourse actions to maximize their total profits. A main question is how the increased profits should be shared among the members of the cooperation. Cooperative game theory mainly studies this issue and proposes the core concept for stability of the cooperation. The core is the set of all stable profit divisions such that no group of players would like to split off from cooperation and form a smaller coalition. We provide sufficient conditions for the associated cooperative games to have non-empty cores. From a similar point of view, several papers studied cooperation in a newsvendor setting to benefit from inventory pooling (see [10–16]). All of these studies showed the core non-emptiness of the associated cooperative game without determining a core element. In a recent study, Montrucchio and Scarsini [17] showed that the core of a simple newsvendor game is non-empty by identifying

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a core element. Chen and Zhang [18] presented an approach using strong duality of stochastic linear programs to identify core elements. Although they illustrated their approach on the games associated with the newsvendor situation with multiple warehouses as introduced by Özen et al. [13], their approach can be applied to a class of cooperative games arising from inventory pooling. The contribution of our paper is two fold:

- We provide a general framework that covers a general class of situations in which the uncertainty can deal with different aspects in the system, e.g., random demand and random yield. By means of the general framework, we provide a set of sufficient conditions, which are relatively easy to check, for core non-emptiness of the associated cooperative games. We remark that all of the studies above consider situations where the uncertainty deals with demand and they all fit into our framework.
- We extend the results in the literature such that our approach can handle any cost and revenue structures in the second stage of the stochastic program as long as the profit functions stay concave whereas the approaches in the literature are mainly based on the linearity assumption on costs and revenues.

Nonemptiness of the core is also investigated in the literature dealing with investments. Borm et al. [19] studied firms' cooperative investments in capital deposits and [20] considered a cooperative investment situation where the firms bundle their resources to invest in long term projects. Both studies assume a deterministic setting. In this paper, we consider a two-stage stochastic variant of these problems as well.

The rest of the paper is organized as follows. In Section 2, we give preliminaries on positively homogeneous functions and cooperative game theory. In Section 3, we introduce a framework for situations with decision making under uncertainty and cooperation possibilities, and we focus on a special class of situations, called stochastic cooperative decision situations. This class captures a broad range of cooperation situations under uncertainty. We show that the cooperative games associated with these situations are totally balanced and, hence, they have non-empty cores. Afterwards, in Section 4, we provide some example situations that can be analyzed in this framework. We conclude the paper with further discussions in Section 5.

2. Preliminaries

In this section, we give preliminaries on positively homogeneous functions and cooperative game theory.

A function f on $\mathbb{R}^n$ is called *positively homogeneous* (of degree 1) if for every $x \in \mathbb{R}^n$ and $\lambda \in (0, \infty)$

$$f(\lambda x) = \lambda f(x).$$

Note that if a function f is positively homogeneous, then $f(0) = 0$. Moreover, all linear functions are positively homogeneous.

Theorem 1. *Let f be a function from $\mathbb{R}^n$ to $\mathbb{R}$. If f is a positively homogeneous concave function, then for every $\lambda_1 \geq 0, \ldots, \lambda_m \geq 0$ and $x_1, \ldots, x_m \in \mathbb{R}^n$*

$$f(\lambda_1 x_1 + \cdots + \lambda_m x_m) \geq \lambda_1 f(x_1) + \cdots + \lambda_m f(x_m). \tag{1}$$

Proof. Without loss of generality, we may assume that $\lambda_i > 0$ for every $i \in \{1, .., m\}$. We have

$$\begin{aligned} f(\lambda_1 x_1 + \cdots + \lambda_m x_m) &= f\left(\sum_{i=1}^{m} \lambda_i x_i\right) \\ &= \sum_{j=1}^{m} \lambda_j \left(f\left(\sum_{i=1}^{m} \frac{\lambda_i}{\sum_{j=1}^{m} \lambda_j} x_i \right)\right) \\ &\geq \sum_{j=1}^{m} \lambda_j \left(\sum_{i=1}^{m} \frac{\lambda_i}{\sum_{j=1}^{m} \lambda_j} f(x_i) \right) \\ &= \sum_{i=1}^{m} \lambda_i f(x_i) \\ &= \lambda_1 f(x_1) + \cdots + \lambda_m f(x_m). \end{aligned}$$

The second equality follows since f is positively homogeneous. The inequality follows from concavity of f. This completes the proof. □

Cooperative game theory deals with situations, where a group of players cooperate by coordinating their actions to obtain a joint profit. It is usually assumed that binding agreements between the players are the mean of the cooperation. A main question of concern is how this profit will be divided among the cooperating players.

Let N be a finite set of players, $N = \{1, \ldots, n\}$. A subset of N is called a *coalition*. A function v, assigning a value $v(S)$ to every coalition $S \subseteq N$ with $v(\emptyset) = 0$, is called a *characteristic function*. The value $v(S)$ is interpreted as the maximum total profit that coalition S can obtain through cooperation. Assuming that the benefit of a coalition S can be transferred between the players of S, a pair (N, v) is called a *cooperative game with transferable utility* (TU-game) or *a game in coalitional form*. For a game (N, v), $S \subset N$ and $S \neq \emptyset$, the *subgame* $(S, v_{|S})$ is defined by $v_{|S}(T) = v(T)$ for each coalition $T \subseteq S$.

In reality, the players are not primarily interested in benefits of a coalition but in their individual benefits that they make out of that coalition. A division is a *payoff vector* $y = (y_i)_{i \in N} \in \mathbb{R}^N$, specifying for each player $i \in N$ the benefit y_i. A division y is called *efficient* if $\sum_{i \in N} y_i = v(N)$ and *individually rational* if $y_i \geq v(\{i\})$ for all $i \in N$. Individual rationality means that every player gets at least as much as what he could obtain by staying alone. The set of all individually rational and efficient divisions constitutes the *imputation set*:

$$I(v) = \left\{ y \in \mathbb{R}^N | \sum_{i \in N} y_i = v(N) \text{ and } y_i \geq v(\{i\}) \text{ for each } i \in N \right\}.$$

If these rationality requirements are extended to all coalitions, we obtain the *core*:

$$Core(v) = \left\{ y \in \mathbb{R}^N | \sum_{i \in N} y_i = v(N) \text{ and } \sum_{i \in S} y_i \geq v(S) \right. \\ \left. \text{for each } S \subseteq N \right\}.$$

Thus, the core consists of all imputations in which no group of players has an incentive to split off from the grand coalition N and form a smaller coalition, because they collectively receive at least as much as what they can obtain by cooperating on their own. Note that the core of a game can be empty.

[21] and [22] independently made a general characterization of games with a non-empty core by the notion of balancedness. Let us define the vector e^S for all $S \subseteq N$ by $e_i^S = 1$ for all $i \in S$ and $e_i^S = 0$ for all $i \in N \setminus S$. A map $\kappa : 2^N \setminus \{\emptyset\} \to [0, 1]$ is called a *balanced map* if $\sum_{S \in 2^N \setminus \{\emptyset\}} \kappa(S) e^S = e^N$. Further, a game (N, v) is called *balanced* if for every balanced map $\kappa : 2^N \setminus \{\emptyset\} \to [0, 1]$ it holds that $\sum_{S \in 2^N \setminus \{\emptyset\}} \kappa(S) v(S) \leq v(N)$. The following theorem is due to [21] and [22].